Gauge invariant perturbations of Petrov type D spacetimes - II

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Gauge invariant perturbations of ~~Petrov type D spacetimes~~ - II

Schwarzschild spacetime

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Introduction

- Regge-Wheeler, Moncrief, Zerilli - they all used harmonic decomposition to find the equations (2nd order in ’t’ and ‘r’) their invariants/variables solve. All this is nicely summarized in Martel-Poisson.
- This will not be possible* to do in Kerr spacetime.
- A warm-up exercise to tackle Kerr - lets not do the harmonic decomposition in Schwarzschild, and derive the 2nd order equation (now a PDE, but, ofcourse, separable) that these or new invariants solve.

* to the best of my knowledge.

Introduction

- The invariants are divided into two categories - usually, spin odd-type, and spin even-type. We do the same, too.
- Odd-type invariants are relatively easy to calculate and are fewer in number. The equations they solve are also easier to derive.
- One of the odd-type invariants is the imaginary part of ψ_2
- The other one we know is its time derivative.
- Nothing simple exists for the even-parity Zerilli invariant.

Odd-type invariants

Imaginary part of ψ_2

$$\mathcal{I} = þ\,ð'\,h_{23} - þ'\,ð'\,h_{13} - þ\,ð\,h_{24} + þ'\,ð\,h_{14}$$

Equation it satisfies is

$$
\begin{aligned}
&8\pi(\rho þ' + \rho' þ - 6\rho\rho')(\rho' ð'\,\mathcal{T}_{13} - \rho' ð\,\mathcal{T}_{14} - \rho ð'\,\mathcal{T}_{23} + \rho ð\,\mathcal{T}_{24}) \\
&+ 8\pi(\rho þ' - \rho' þ)(\rho' ð'\,\mathcal{T}_{13} - \rho' ð\,\mathcal{T}_{14} + \rho ð'\,\mathcal{T}_{23} - \rho ð\,\mathcal{T}_{24}) \\
&= \\
&\Big(\tfrac{1}{2}(\rho þ' + \rho' þ)(\rho þ' + \rho' þ) - \tfrac{1}{2}(\rho þ' - \rho' þ)(\rho þ' - \rho' þ) \\
&- 7\rho\rho'(\rho þ' + \rho' þ) - \rho\rho'(ðð' + ð'ð - 12\rho\rho' - 12\psi_2)\Big)\mathcal{I}
\end{aligned}
$$

Preliminaries

Imaginary part of ψ_2

$$\mathcal{I} = \text{þ}\,\eth'\,h_{23} - \text{þ}'\,\eth'\,h_{13} - \text{þ}\,\eth\,h_{24} + \text{þ}'\,\eth\,h_{14}$$

$$X_{(i)(j)} = X_{\alpha\beta}\,e^{\alpha}_{(i)}\,e^{\beta}_{(j)}$$

$$e^{\alpha}_{(1)} = l^{\alpha}$$

$$e^{\alpha}_{(2)} = n^{\alpha}$$

$$e^{\alpha}_{(3)} = m^{\alpha}$$

$$e^{\alpha}_{(4)} = \bar{m}^{\alpha}$$

$8\pi(\rho\text{þ}' +$ … $\rho\eth'\,\mathcal{T}_{23} + \rho\eth\,\mathcal{T}_{24})$

$+\,8\pi(\rho\text{þ}'$ … $\mathcal{T}_{23} - \rho\eth\,\mathcal{T}_{24})$

$=$

$\Big(\frac{1}{2}(\rho\text{þ}' +$ … $\rho\text{þ}' - \rho'\text{þ})$

$-\,7\rho\rho'(\rho$ … $' - 12\psi_2)\Big)\mathcal{I}$

Preliminaries

Imaginary part of ψ_2

$$\mathcal{I} = \boxed{\text{þ}}\,\eth'\, h_{23} - \text{þ}'\,\eth'\, h_{13} - \boxed{\text{þ}}\,\eth\, h_{24} + \text{þ}'\,\eth\, h_{14}$$

Equation it satisfies is

$$\boxed{\sim l^{\alpha}\partial_{\alpha}}$$

$$8\pi(\rho\text{þ}' + \rho'\text{þ} - 6\rho\rho')(\rho \ldots - \rho'\eth\,\mathcal{T}_{14} - \rho\eth'\,\mathcal{T}_{23} + \rho\eth\,\mathcal{T}_{24})$$
$$+ 8\pi(\rho\text{þ}' - \rho'\text{þ})(\rho'\eth'\,\mathcal{T}_{13} \ldots \rho'\eth\,\mathcal{T}_{14} + \rho\eth'\,\mathcal{T}_{23} - \rho\eth\,\mathcal{T}_{24})$$
$$=$$
$$\Big(\tfrac{1}{2}(\rho\text{þ}' + \rho'\text{þ})(\rho\text{þ}' + \rho'\text{þ}) - \tfrac{1}{2}(\rho\text{þ}' - \rho'\text{þ})(\rho\text{þ}' - \rho'\text{þ})$$
$$- 7\rho\rho'(\rho\text{þ}' + \rho'\text{þ}) - \rho\rho'(\eth\eth' + \eth'\eth - 12\rho\rho' - 12\psi_2)\Big)\mathcal{I}$$

Preliminaries

Imaginary part of ψ_2

$$\mathcal{I} = \text{þ}\,\eth'\,h_{23} - \boxed{\text{þ}'}\,\eth'\,h_{13} - \text{þ}\,\eth\,h_{24} + \boxed{\text{þ}'}\,\eth\,h_{14}$$

Equation it satisfies is

$\sim n^\alpha \partial_\alpha$

$$8\pi(\rho\text{þ}' + \rho'\text{þ} - 6\rho\rho')(\rho'\eth' \cdots \mathcal{T}_{14} - \rho\eth'\,\mathcal{T}_{23} + \rho\eth\,\mathcal{T}_{24})$$
$$+\, 8\pi(\rho\text{þ}' - \rho'\text{þ})(\rho'\eth'\,\mathcal{T}_{13} \cdots \mathcal{T}_{14} + \rho\eth'\,\mathcal{T}_{23} - \rho\eth\,\mathcal{T}_{24})$$
$$=$$
$$\Big(\tfrac{1}{2}(\rho\text{þ}' + \rho'\text{þ})(\rho\text{þ}' + \rho'\text{þ}) - \tfrac{1}{2}(\rho\text{þ}' - \rho'\text{þ})(\rho\text{þ}' - \rho'\text{þ})$$
$$- 7\rho\rho'(\rho\text{þ}' + \rho'\text{þ}) - \rho\rho'(\eth\eth' + \eth'\eth - 12\rho\rho' - 12\psi_2)\Big)\mathcal{I}$$

Preliminaries

Imaginary part of ψ_2

$$\mathcal{I} = \text{þ}\,\eth'\,h_{23} - \text{þ}'\,\eth'\,h_{13} - \text{þ}\,\eth\,h_{24} + \text{þ}'\,\eth\,h_{14}$$

$\sim m^\alpha \partial_\alpha$

Equation it satisfies is

$$8\pi(\rho\text{þ}' + \rho'\text{þ} - 6\rho\rho')(\rho'\eth'\,\mathcal{T}_{13} - \rho'\eth \ldots \mathcal{T}_{23} + \rho\eth\,\mathcal{T}_{24})$$
$$+\,8\pi(\rho\text{þ}' - \rho'\text{þ})(\rho'\eth'\,\mathcal{T}_{13} - \rho'\eth\,\mathcal{T}_{14} + \rho\eth'\,\mathcal{T}_{23} - \rho\eth\,\mathcal{T}_{24})$$
$$=$$
$$\Big(\tfrac{1}{2}(\rho\text{þ}' + \rho'\text{þ})(\rho\text{þ}' + \rho'\text{þ}) - \tfrac{1}{2}(\rho\text{þ}' - \rho'\text{þ})(\rho\text{þ}' - \rho'\text{þ})$$
$$-\,7\rho\rho'(\rho\text{þ}' + \rho'\text{þ}) - \rho\rho'(\eth\eth' + \eth'\eth - 12\rho\rho' - 12\psi_2)\Big)\mathcal{I}$$

Preliminaries

Imaginary part of ψ_2

$$\mathcal{I} = \text{þ}\,\boxed{\eth'}\,h_{23} - \text{þ}'\,\boxed{\eth'}\,h_{13} - \text{þ}\,\eth\,h_{24} + \text{þ}'\,\eth\,h_{14}$$

Equation it satisfies is

$$\boxed{\sim \bar{m}^{\alpha}\partial_{\alpha}}$$

$$8\pi(\rho\text{þ}' + \rho'\text{þ} - 6 \cdots \mathcal{T}_{13} - \rho'\eth\,\mathcal{T}_{14} - \rho\eth'\,\mathcal{T}_{23} + \rho\eth\,\mathcal{T}_{24})$$
$$+\, 8\pi(\rho\text{þ}' - \rho'\text{þ})(\rho'\eth'\,\mathcal{T}_{13} - \rho'\eth\,\mathcal{T}_{14} + \rho\eth'\,\mathcal{T}_{23} - \rho\eth\,\mathcal{T}_{24})$$
$$=$$
$$\Big(\tfrac{1}{2}(\rho\text{þ}' + \rho'\text{þ})(\rho\text{þ}' + \rho'\text{þ}) - \tfrac{1}{2}(\rho\text{þ}' - \rho'\text{þ})(\rho\text{þ}' - \rho'\text{þ})$$
$$- 7\rho\rho'(\rho\text{þ}' + \rho'\text{þ}) - \rho\rho'(\eth\eth' + \eth'\eth - 12\rho\rho' - 12\psi_2)\Big)\mathcal{I}$$

Preliminaries

Imaginary part of ψ_2

$$\mathcal{I} = \text{þ}\,\eth'\,h_{23} - \text{þ}'\,\eth'\,h_{13} - \text{þ}\,\eth\,h_{24} + \text{þ}'\,\eth\,h_{14}$$

$\sim \partial_{r_*}\partial_{r_*}$

Ec … sfies is

$$8\pi(\rho\text{þ}' + \rho'\text{þ} - 6\rho\rho')(\rho'\eth'\,\mathcal{T}_{13} - \rho'\eth\,\mathcal{T}_{14} - \rho\eth'\,\mathcal{T}_{23} + \rho\eth\,\mathcal{T}_{24})$$
$$+ 8\pi(\rho\text{þ}' - \rho'\text{þ})(\rho'\eth'\,\mathcal{T}_{13} - \rho'\eth\,\mathcal{T}_{14} + \rho\eth'\,\mathcal{T}_{23} - \rho\eth\,\mathcal{T}_{24})$$
$$=$$
$$\Big(\tfrac{1}{2}\boxed{(\rho\text{þ}' + \rho'\text{þ})(\rho\text{þ}' + \rho'\text{þ})} - \tfrac{1}{2}(\rho\text{þ}' - \rho'\text{þ})(\rho\text{þ}' - \rho'\text{þ})$$
$$- 7\rho\rho'(\rho\text{þ}' + \rho'\text{þ}) - \rho\rho'(\eth\eth' + \eth'\eth - 12\rho\rho' - 12\psi_2)\Big)\mathcal{I}$$

Preliminaries

Imaginary part of ψ_2

$$\mathcal{I} = \text{þ}\,\eth'\,h_{23} - \text{þ}'\,\eth'\,h_{13} - \text{þ}\,\eth\,h_{24} + \text{þ}'\,\eth\,h_{14}$$

Ec $\sim \partial_t \partial_t$ sfies is

$$8\pi(\rho\text{þ}' + \rho'\text{þ} - 6\rho\rho')(\rho'\eth'\,\mathcal{T}_{13} - \rho'\eth\,\mathcal{T}_{14} - \rho\eth'\,\mathcal{T}_{23} + \rho\eth\,\mathcal{T}_{24})$$
$$+ 8\pi(\rho\text{þ}' - \rho'\text{þ})(\rho'\eth'\,\mathcal{T}_{13} - \rho'\eth\,\mathcal{T}_{14} + \rho\eth'\,\mathcal{T}_{23} - \rho\eth\,\mathcal{T}_{24})$$
$$=$$
$$\Big(\tfrac{1}{2}(\rho\text{þ}' + \rho'\text{þ})(\rho\text{þ}' + \rho'\text{þ}) - \tfrac{1}{2}\boxed{(\rho\text{þ}' - \rho'\text{þ})(\rho\text{þ}' - \rho'\text{þ})}$$
$$- 7\rho\rho'(\rho\text{þ}' + \rho'\text{þ}) - \rho\rho'(\eth\eth' + \eth'\eth - 12\rho\rho' - 12\psi_2)\Big)\mathcal{I}$$

Preliminaries

Imaginary part of ψ_2

$$\mathcal{I} = \text{þ}\,\eth'\,h_{23} - \text{þ}'\,\eth'\,h_{13} - \text{þ}\,\eth\,h_{24} + \text{þ}'\,\eth\,h_{14}$$

Equation it satisfies is

$$\sim \ell(\ell+1)$$

$$8\pi(\rho\text{þ}' + \ \ldots \ \mathcal{T}_{13} - \rho'\eth\,\mathcal{T}_{14} - \rho\eth'\,\mathcal{T}_{23} + \rho\eth\,\mathcal{T}_{24})$$
$$+ 8\pi(\rho\text{þ}' - \rho'\text{þ})(\rho'\eth'\,\mathcal{T}_{13} - \rho'\eth\,\mathcal{T}_{14} + \rho\eth'\,\mathcal{T}_{23} - \rho\eth\,\mathcal{T}_{24})$$
$$=$$
$$\Big(\tfrac{1}{2}(\rho\text{þ}' + \rho'\text{þ})(\rho\text{þ}' + \rho'\text{þ}) - \tfrac{1}{2}(\rho\text{þ}' - \rho'\text{þ})(\rho\text{þ}' - \rho'\text{þ})$$
$$- 7\rho\rho'(\rho\text{þ}' + \rho'\text{þ}) - \rho\rho'\boxed{(\eth\eth' + \eth'\eth - 12\rho\rho' - 12\psi_2)}\Big)\mathcal{I}$$

Odd-type invariants

Imaginary part of ψ_2

$$\mathcal{I} = þ\,ð'\,h_{23} - þ'\,ð'\,h_{13} - þ\,ð\,h_{24} + þ'\,ð\,h_{14}$$

Equation it satisfies is

$$\begin{aligned}
&8\pi(\rho þ' + \rho' þ - 6\rho\rho')(\rho' ð'\,\mathcal{T}_{13} - \rho' ð\,\mathcal{T}_{14} - \rho ð'\,\mathcal{T}_{23} + \rho ð\,\mathcal{T}_{24}) \\
&+ 8\pi(\rho þ' - \rho' þ)(\rho' ð'\,\mathcal{T}_{13} - \rho' ð\,\mathcal{T}_{14} + \rho ð'\,\mathcal{T}_{23} - \rho ð\,\mathcal{T}_{24}) \\
&= \\
&\Big(\tfrac{1}{2}(\rho þ' + \rho' þ)(\rho þ' + \rho' þ) - \tfrac{1}{2}(\rho þ' - \rho' þ)(\rho þ' - \rho' þ) \\
&- 7\rho\rho'(\rho þ' + \rho' þ) - \rho\rho'(ðð' + ð'ð - 12\rho\rho' - 12\psi_2)\Big)\mathcal{I}
\end{aligned}$$

Odd-type invariants

Time derivative of the imaginary part of ψ_2

$$\mathcal{I}_t = (\eth\eth' + \eth'\eth - 4\rho\rho' - 4\psi_2)\,(\rho'\eth' h_{13} - \rho'\eth h_{14} + \rho\eth' h_{23} - \rho\eth h_{24}) - \left(\rho\text{þ}' + \rho'\text{þ} - 4\rho\rho'\right)(\eth'\eth' h_{33} - \eth\eth h_{44})$$

$\mathcal{I}_t$ → ψ_{RW} in Bernard's talk

The relation between the two

$$\mathcal{I}_t = (\rho\text{þ}' - \rho'\text{þ})\mathcal{I} + 2\rho\,\eth'\,\mathcal{E}_{23} - 2\rho\,\eth\,\mathcal{E}_{24} + 2\rho'\,\eth'\,\mathcal{E}_{13} - 2\rho'\,\eth\,\mathcal{E}_{14}$$

Odd-type invariants

Time derivative of the imaginary part of ψ_2

$$\mathcal{I}_t = (\eth\eth' + \eth'\eth - 4\rho\rho' - 4\psi_2)\,(\rho'\eth' h_{13} - \rho'\eth h_{14} + \rho\eth' h_{23} - \rho\eth h_{24}) - \left(\rho þ' + \rho' þ - 4\rho\rho'\right)(\eth'\eth' h_{33} - \eth\eth h_{44})$$

Equation it satisfies is

$$8\pi(\rho þ' + \rho' þ - 8\rho\rho')(\rho þ' - \rho' þ)(\rho'\eth' \mathcal{T}_{13} - \rho'\eth \mathcal{T}_{14} - \rho\eth' \mathcal{T}_{23} + \rho\eth \mathcal{T}_{24})$$
$$+ 8\pi(\rho þ' + \rho' þ - 18\rho\rho')(\rho þ' + \rho' þ)(\rho'\eth' \mathcal{T}_{13} - \rho'\eth \mathcal{T}_{14} + \rho\eth' \mathcal{T}_{23} - \rho\eth \mathcal{T}_{24})$$
$$- 16\pi\rho\rho'(\eth\eth' + \eth'\eth - 24\rho\rho' - 14\psi_2)(\rho'\eth' \mathcal{T}_{13} - \rho'\eth \mathcal{T}_{14} + \rho\eth' \mathcal{T}_{23} - \rho\eth \mathcal{T}_{24})$$
$$=$$
$$\Big(\tfrac{1}{2}(\rho þ' + \rho' þ)(\rho þ' + \rho' þ) - \tfrac{1}{2}(\rho þ' - \rho' þ)(\rho þ' - \rho' þ)$$
$$- 9\rho\rho'(\rho þ' + \rho' þ) - \rho\rho'(\eth\eth' + \eth'\eth - 24\rho\rho' - 14\psi_2)\Big)\mathcal{I}_t$$

Even-type invariants

Zerilli's even-parity invariant

$$
\begin{aligned}
\mathcal{I}_z = &\left(\eth\eth' + \eth'\eth - 4\rho\rho' - 4\psi_2\right)\left(\eth\eth' + \eth'\eth\right)\left(\rho\text{þ}' - \rho'\text{þ}\right) h_{34} \\
&+ \left(\eth\eth' + \eth'\eth - 4\rho\rho' - 4\psi_2\right)\left(\eth\eth' + \eth'\eth\right)\left({\rho'}^2 h_{11} - \rho^2 h_{22}\right) \\
&- \left(\eth\eth' + \eth'\eth - 4\rho\rho' + 2\psi_2\right)\left(\rho\text{þ}' - \rho'\text{þ}\right)\left(\eth'\eth' h_{33} + \eth\eth h_{44}\right) \\
&+ 2(2\rho\rho' - \psi_2)\left(\eth\eth' + \eth'\eth - 4\rho\rho' - 4\psi_2\right)\left(\rho'\eth' h_{13} + \rho'\eth h_{14} - \rho\eth' h_{23} - \rho\eth h_{24}\right) \\
&- 2\rho\rho'\left(\eth\eth' + \eth'\eth - 4\rho\rho' - 4\psi_2\right)\left(\text{þ}'\eth' h_{13} + \text{þ}'\eth h_{14} - \text{þ}\eth' h_{23} - \text{þ}\eth h_{24}\right)
\end{aligned}
$$

5th order

Even-type invariants

$$
\begin{aligned}
&8\pi\rho\rho'(\eth\eth' + \eth'\eth - 4\rho\rho' - 4\psi_2)(\eth\eth' + \eth'\eth - 4\rho\rho' + 2\psi_2)^2(\rho\text{þ}' + \rho'\text{þ})\\
&\times(\rho'\eth'\,\mathcal{T}_{13} + \rho'\eth\,\mathcal{T}_{14} - \rho\eth'\,\mathcal{T}_{23} - \rho\eth\,\mathcal{T}_{24})\\
&- 2\pi(\eth\eth' + \eth'\eth)(\eth\eth' + \eth'\eth - 4\rho\rho' - 4\psi_2)(\eth\eth' + \eth'\eth - 4\rho\rho' + 2\psi_2)^2(\rho\text{þ}' + \rho'\text{þ})\\
&\times(\rho'^2\,\mathcal{T}_{11} - \rho^2\,\mathcal{T}_{22})\\
&- 2\pi(\eth\eth' + \eth'\eth)(\eth\eth' + \eth'\eth - 4\rho\rho' - 4\psi_2)(\eth\eth' + \eth'\eth - 4\rho\rho' + 2\psi_2)^2(\rho\text{þ}' + \rho'\text{þ})\\
&\times(\rho'^2\,\mathcal{T}_{11} + 2\rho\rho'\,\mathcal{T}_{12} + \rho^2\,\mathcal{T}_{22})\\
&+ 8\pi\rho\rho'(\eth\eth' + \eth'\eth - 4\rho\rho' + 2\psi_2)^3(\rho\text{þ}' + \rho'\text{þ})\,(\eth'\eth'\,\mathcal{T}_{33} + \eth\eth\,\mathcal{T}_{44})\\
&+ 8\pi\rho\rho'(\eth\eth' + \eth'\eth - 4\rho\rho' - 4\psi_2)(\eth\eth' + \eth'\eth - 4\rho\rho' + 2\psi_2)^2(\rho\text{þ}' - \rho'\text{þ})\\
&\times(\rho'\eth'\,\mathcal{T}_{13} + \rho'\eth\,\mathcal{T}_{14} + \rho\eth'\,\mathcal{T}_{23} + \rho\eth\,\mathcal{T}_{24})\\
&+ 8\pi\rho\rho'(\eth\eth' + \eth'\eth)(\eth\eth' + \eth'\eth - 4\rho\rho' - 4\psi_2)(\eth\eth' + \eth'\eth - 4\rho\rho' + 2\psi_2)\\
&\times(\eth\eth' + \eth'\eth - 4\rho\rho' + 8\psi_2)\,(\rho'^2\,\mathcal{T}_{11} - \rho^2\,\mathcal{T}_{22})\\
&+ 4\pi(\eth\eth' + \eth'\eth - 4\rho\rho' - 4\psi_2)(\eth\eth' + \eth'\eth - 4\rho\rho' + 2\psi_2)\\
&\times\Big((\eth\eth' + \eth'\eth)^2 + 2(\psi_2 + 8\rho\rho')(\eth\eth' + \eth'\eth) - 8(\psi_2^2 + 7\rho\rho'\psi_2 - 6\rho^2\rho'^2)\Big)\\
&\times(\rho'\eth'\,\mathcal{T}_{13} + \rho'\eth\,\mathcal{T}_{14} - \rho\eth'\,\mathcal{T}_{23} - \rho\eth\,\mathcal{T}_{24})\\
&=\\
&\tfrac{1}{4}(\eth\eth' + \eth'\eth - 4\rho\rho' + 2\psi_2)^2\big[(\rho\text{þ}' + \rho'\text{þ})^2 - (\rho\text{þ}' - \rho'\text{þ})^2\big]\mathcal{I}_z\\
&- \tfrac{1}{2}\rho\rho'(\eth\eth' + \eth'\eth - 4\rho\rho' + 2\psi_2)\big(13(\eth\eth' + \eth'\eth) + 38\psi_2 - 52\rho\rho'\big)\mathcal{I}_z\\
&- \tfrac{1}{2}\rho\rho'(\eth\eth' + \eth'\eth - 4\rho\rho' + 2\psi_2)\\
&\times\big((\eth\eth' + \eth'\eth)^2 - 16(\eth\eth' + \eth'\eth)(\psi_2 + 4\rho\rho') - 24(\psi_2 - \rho\rho')(\psi_2 + 10\rho\rho')\big)\mathcal{I}_z
\end{aligned}
$$

Even-type invariants

$$
\begin{aligned}
&8\pi\rho\rho'(\eth\eth' + \eth'\eth - 4\rho\rho' - 4\psi_2)(\eth\eth' + \eth'\eth - 4\rho\rho' + 2\psi_2)^2(\rho\text{þ}' \,+\, \rho'\text{þ})\\
&\times\,(\rho'\eth'\,\mathcal{T}_{13} + \rho'\eth\,\mathcal{T}_{14} - \rho\eth'\,\mathcal{T}_{23} - \rho\eth\,\mathcal{T}_{24})\\
&-2\pi(\eth\eth' + \eth'\eth)(\eth\eth' + \eth'\eth - 4\rho\rho' - 4\psi_2)(\eth\eth' + \eth'\eth - 4\rho\rho' + 2\psi_2)^2(\rho\text{þ}' \,+\, \rho'\text{þ})\\
&\times\,({\rho'}^2\,\mathcal{T}_{11} - \rho^2\,\mathcal{T}_{22})\\
&-2\pi(\eth\eth' + \eth'\eth)(\eth\eth' + \eth'\eth - 4\rho\rho' - 4\psi_2)(\eth\eth' + \eth'\eth - 4\rho\rho' + 2\psi_2)^2(\rho\text{þ}' \,+\, \rho'\text{þ})\\
&\times\,({\rho'}^2\,\mathcal{T}_{11} + 2\rho\rho'\,\mathcal{T}_{12} + \rho^2\,\mathcal{T}_{22})\\
&+8\pi\rho\rho'(\eth\eth' + \eth'\eth - 4\rho\rho' + 2\psi_2)^3(\rho\text{þ}' \,+\, \rho'\text{þ})\,(\eth'\eth'\,\mathcal{T}_{33} + \eth\eth\,\mathcal{T}_{44})\\
&+8\pi\rho\rho'(\eth\eth' + \eth'\eth - 4\rho\rho' - 4\psi_2)(\eth\eth' + \eth'\eth - 4\rho\rho' + 2\psi_2)^2(\rho\text{þ}' \,-\, \rho'\text{þ})\\
&\times\,(\rho'\eth'\,\mathcal{T}_{13} + \rho'\eth\,\mathcal{T}_{14} + \rho\eth'\,\mathcal{T}_{23} + \rho\eth\,\mathcal{T}_{24})\\
&+8\pi\rho\rho'(\eth\eth' + \eth'\eth)(\eth\eth' + \eth'\eth - 4\rho\rho' - 4\psi_2)(\eth\eth' + \eth'\eth - 4\rho\rho' + 2\psi_2)\\
&\times\,(\eth\eth' + \eth'\eth - 4\rho\rho' + 8\psi_2)\,({\rho'}^2\,\mathcal{T}_{11} - \rho^2\,\mathcal{T}_{22})\\
&+4\pi(\eth\eth' + \eth'\eth - 4\rho\rho' - 4\psi_2)(\eth\eth' + \eth'\eth - 4\rho\rho' + 2\psi_2)\\
&\times\,\Big((\eth\eth' + \eth'\eth)^2 + 2(\psi_2 + 8\rho\rho')(\eth\eth' + \eth'\eth) - 8(\psi_2^2 + 7\rho\rho'\psi_2 - 6\rho^2{\rho'}^2)\Big)\\
&\times\,(\rho'\eth'\,\mathcal{T}_{13} + \rho'\eth\,\mathcal{T}_{14} - \rho\eth'\,\mathcal{T}_{23} - \rho\eth\,\mathcal{T}_{24})\\
&=\\
&\tfrac{1}{4}(\eth\eth' + \eth'\eth - 4\rho\rho' + 2\psi_2)^2\,\Big[\boxed{(\rho\text{þ}' \,+\, \rho'\text{þ})^2 - (\rho\text{þ}' \,-\, \rho'\text{þ})^2}\Big]\,\mathcal{I}_z\\
&-\tfrac{1}{2}\rho\rho'(\eth\eth' + \eth'\eth - 4\rho\rho' + 2\psi_2)\big(13(\eth\eth' + \eth'\eth) + 38\psi_2 - 52\rho\rho'\big)\mathcal{I}_z\\
&-\tfrac{1}{2}\rho\rho'(\eth\eth' + \eth'\eth - 4\rho\rho' + 2\psi_2)\\
&\times\,\big((\eth\eth' + \eth'\eth)^2 - 16(\eth\eth' + \eth'\eth)(\psi_2 + 4\rho\rho') - 24(\psi_2 - \rho\rho')(\psi_2 + 10\rho\rho')\big)\mathcal{I}_z
\end{aligned}
$$

Even-type invariants

New even-parity invariant

$$\mathcal{I}_{E1} = (\eth\eth' + \eth'\eth - 4\rho\rho' - 4\psi_2)\Bigg(\rho'^2 h_{11} + 2\rho\rho' h_{12} + \rho^2 h_{22} + \rho'\eth' h_{13} + \rho'\eth h_{14} + \rho\eth' h_{23} + \rho\eth h_{24}$$
$$- \left[\rho\text{þ}' + \rho'\text{þ} + \frac{1}{2}\eth\eth' + \frac{1}{2}\eth'\eth - 2\rho\rho' + \psi_2\right] h_{34} + \frac{1}{2}\eth'\eth' h_{33} + \frac{1}{2}\eth\eth h_{44}\Bigg) + 3\psi_2\left(\eth'\eth' h_{33} + \eth\eth h_{44}\right)$$

4th order

Even-type invariants

$$
\begin{aligned}
&8\pi(\eth\eth' + \eth'\eth - 4\rho\rho' + 2\psi_2)(\eth\eth' + \eth'\eth - 4\rho\rho' - 4\psi_2)(\eth\eth' + \eth'\eth) \\
&\quad \times (\rho'^2 \mathcal{T}_{11} + 2\rho\rho'\mathcal{T}_{12} + \rho^2\mathcal{T}_{22}) \\
&-32\pi\rho\rho'(\eth\eth' + \eth'\eth - 4\rho\rho' + 2\psi_2)^2(\eth'\eth'\mathcal{T}_{33} + \eth\eth\mathcal{T}_{44}) \\
&-16\pi(\eth\eth' + \eth'\eth - 4\rho\rho' + 2\psi_2)(\eth\eth' + \eth'\eth - 4\rho\rho' - 4\psi_2)(\rho\text{þ}' - \rho'\text{þ}) \\
&\quad \times (\rho'^2\mathcal{T}_{11} - \rho^2\mathcal{T}_{22}) \\
&-32\pi\rho\rho'(\eth\eth' + \eth'\eth - 4\rho\rho' + 2\psi_2)(\eth\eth' + \eth'\eth - 4\rho\rho' - 4\psi_2) \\
&\quad \times (\rho'\eth'\mathcal{T}_{13} + \rho'\eth\mathcal{T}_{14} + \rho\eth'\mathcal{T}_{23} + \rho\eth\mathcal{T}_{24}) \\
&-16\pi(\eth\eth' + \eth'\eth - 4\rho\rho' + 2\psi_2)(\eth\eth' + \eth'\eth - 4\rho\rho' - 4\psi_2)(\rho\text{þ}' + \rho'\text{þ}) \\
&\quad \times (\rho'^2\mathcal{T}_{11} - 2\rho\rho'\mathcal{T}_{12} - \rho^2\mathcal{T}_{22}) \\
&-8\pi\left[(\eth\eth' + \eth'\eth)^2 + (\eth\eth' + \eth'\eth)(2\psi_2 - 20\rho\rho') + 64\rho^2\rho'^2 - 80\psi_2\rho\rho'\right] \\
&\quad \times (\eth\eth' + \eth'\eth - 4\rho\rho' - 4\psi_2)(\rho'^2\mathcal{T}_{11} - 2\rho\rho'\mathcal{T}_{12} - \rho^2\mathcal{T}_{22}) \\
&= \\
&2(\eth\eth' + \eth'\eth - 4\rho\rho' + 2\psi_2)\boxed{\left[(\rho\text{þ}' + \rho'\text{þ})^2 - (\rho\text{þ}' - \rho'\text{þ})^2\right]}\mathcal{I}_{E1} \\
&-4\rho\rho'(11\eth\eth' + 11\eth'\eth - 44\rho\rho' + 34\psi_2)(\rho\text{þ}' + \rho'\text{þ})\mathcal{I}_{E1} \\
&-4\rho\rho'\left[(\eth\eth' + \eth'\eth)^2 - 2(\eth\eth' + \eth'\eth)(7\psi_2 + 22\rho\rho') - 20(\psi_2^2 + 8\rho\rho'\psi_2 - 8\rho^2\rho'^2)\mathcal{I}_{E1}\right]
\end{aligned}
$$

Invariants

- Until now we had 2 odd-parity invariants, and an even parity invariant.
- We also had the “2nd order” PDE these three solved.
- And now, we have a new even parity invariant, and the “2nd order” PDE it solves.
- This brings the list to 2 odd-parity and 2 even-parity invariants in Schwarzschild spacetime.
- We later had another odd-parity invariant but didn’t look for the equation it solves yet.

Invariants

- One then wonders - are there more invariants?
- If so, how does one find them? Is there a recipe?
- Yes, there is… We use a combination of Regge-Wheeler's and Steve Detweiler's ideas.
- Just like RW, we write the different "components" of the metric perturbation and look at their gauge-contribution.
- Just like SD, we look for different possible combinations to eliminate the gauge-contributions.
- All this without the separation into radial and angular parts,

GHP-form

Lets look at the odd spin-type case..

The gauge-change for the following "components" are

$$\eth'\eth' h_{33} - \eth\eth h_{44} \longrightarrow (\eth\eth' + \eth'\eth - 4\rho\rho' - 4\psi_2)(\eth'\xi_3 - \eth\xi_4)$$

$$\rho'\eth' h_{13} - \rho'\eth h_{14} + \rho\eth' h_{23} - \rho\eth h_{24} \longrightarrow (\rho\text{þ}' + \rho'\text{þ})(\eth'\xi_3 - \eth\xi_4)$$

$$\rho'\eth' h_{13} - \rho'\eth h_{14} - \rho\eth' h_{23} + \rho\eth h_{24} \longrightarrow -(\rho\text{þ}' - \rho'\text{þ})(\eth'\xi_3 - \eth\xi_4)$$

And so you have three possible ways to eliminate the gauge vector, hence three odd invariants.

GHP-form

The odd-spin type invariants are

$$(\rho \text{þ}' - \rho' \text{þ})(\eth' \eth' h_{33} - \eth \eth h_{44}) + (\eth \eth' + \eth' \eth - 4\rho\rho' - 4\psi_2)(\rho' \eth' h_{13} - \rho' \eth h_{14} - \rho \eth' h_{23} + \rho \eth h_{24})$$

$$(\rho \text{þ}' + \rho' \text{þ} - 4\rho\rho')(\eth' \eth' h_{33} - \eth \eth h_{44}) - (\eth \eth' + \eth' \eth - 4\rho\rho' - 4\psi_2)(\rho' \eth' h_{13} - \rho' \eth h_{14} + \rho \eth' h_{23} - \rho \eth h_{24})$$

$$(\rho \text{þ}' - \rho' \text{þ})(\rho' \eth' h_{13} - \rho' \eth h_{14} + \rho \eth' h_{23} - \rho \eth h_{24}) + (\rho \text{þ}' + \rho' \text{þ} - 2\rho\rho')(\rho' \eth' h_{13} - \rho' \eth h_{14} - \rho \eth' h_{23} + \rho \eth h_{24})$$

Time derivative of the imaginary part of ψ_2

Imaginary part of ψ_2

Radial derivative of the imaginary part of ψ_2

We now have PDEs that each one of these solve!

GHP-form

The odd-spin type invariants are

$$(\rho þ' - \rho' þ)(ð'ð' h_{33} - ðð h_{44}) + (ðð' + ð'ð - 4\rho\rho' - 4\psi_2)(\rho' ð' h_{13} - \rho' ð h_{14} - \rho ð' h_{23} + \rho ð h_{24})$$

$$(\rho þ' + \rho' þ - 4\rho\rho')(ð'ð' h_{33} - ðð h_{44}) - (ðð' + ð'ð - 4\rho\rho' - 4\psi_2)(\rho' ð' h_{13} - \rho' ð h_{14} + \rho ð' h_{23} - \rho ð h_{24})$$

$$(\rho þ' - \rho' þ)(\rho' ð' h_{13} - \rho' ð h_{14} + \rho ð' h_{23} - \rho ð h_{24}) + (\rho þ' + \rho' þ - 2\rho\rho')(\rho' ð' h_{13} - \rho' ð h_{14} - \rho ð' h_{23} + \rho ð h_{24})$$

One then goes on, and systematically finds the even spin type invariants. Its more complicated to show the components or the invariants here.

Finally, we end up with thirteen even-type invariants using this recipe, and three odd-type invariants.

Summary

- Used GHP-tools to calculate the already known odd- and even-type invariants, and derived the equation they solve. All this, without separation.
- Calculated a new 4th order even-type invariant, and derived the equation it solves.
- Calculated a new 4th order odd-type invariant.
- Used a systematic approach to find other invariants and ended up with 3 odd-type and 13* even-type invariants.

* though linearly independent a few of them can be related to each other using Einstein tensors and derivatives

Summary

- Taking this to Kerr space-time — even the first steps are very complicated given that operators and variables that "commuted" in Schwarzschild no more commute with each other in Kerr spacetime — every formula has a lot of "unwanted" baggage.
- How many invariants will we get in Kerr (if any possible)?
- Will it be possible to get the PDE those invariants solve?