# Progress Toward Post-Adiabatic EMRI Waveforms

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# Status of EMRI Computations

- Self-force computations (worldline evolution)
  - A great deal of activity, many useful regularization schemes
  - Community is close to first order self-force for generic orbits
- Computation of waveforms
  - Adiabatic order: Analytic formalism in place, numerical implementation [Burko, Lackeos 2012]
  - Post-adiabatic order: Analytic formalism and foundations in progress [This talk] - extends [Flanagan, Hinderer 2008; Pound 2015]

| | Adiabatic | Post-adiabatic |
|---|---|---|
| Required Order of Self-Force | First Order Dissipative | Second Order Dissipative + First Order Conservative |
| Errors in Amplitude of Waveform | $\mathcal{O}(\epsilon)$ | $\mathcal{O}(\epsilon^2)$ |
| Errors in Phase of Waveform | $\mathcal{O}(1)$ | $\mathcal{O}(\epsilon)$ |

# Foundation:Two-Timescale for the Interaction Region

- Two timescale approximation promotes time dependence to multiple (temporarily) independent variables $t \to \{\tilde{t}, q^A\}$

$$\tilde{t} = \frac{\mu}{M} t \equiv \epsilon t \qquad \frac{dq^A}{dt} = \Omega(\tilde{t}, \epsilon)$$

- Conserved quantities $J^M$ closely related to momentum components - direct in the case of $E$, $L_z$
- Action angle variables $q^A$ coordinates on compact directions of the symplectic manifold
- Periodic behavior depends on $q^A$, secular depends on $\tilde{t}$
- Worldline can be expressed using action angle variables :

$$\frac{dJ^M}{dt} = \epsilon G^{(1)M}(J^{(0)M}(\tilde{t}), q^A) + \mathcal{O}(\epsilon^2)$$
$$\frac{dq^A}{dt} = \Omega^A(J^{(0)M}(\tilde{t})) + \epsilon g^{(1)A}(J^{(0)M}(\tilde{t}), q^A)$$

# Improved Long-Time Fidelity

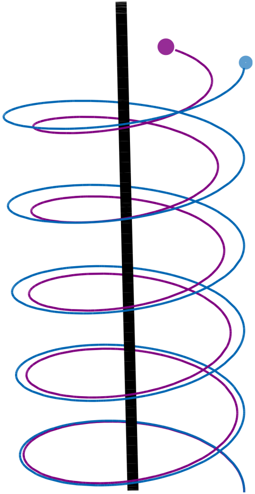
- Metric ansatz ($g^{(0)}_{\alpha\beta}$ taken to be Schwarzschild)

$$g_{\alpha\beta} = g^{(0)}_{\alpha\beta}(\bar{x}^i) + \epsilon h^{(1)}_{\alpha\beta}(\tilde{t}, q^A, \bar{x}^i) + \epsilon^2 h^{(1)}_{\alpha\beta}(\tilde{t}, q^A, \bar{x}^i) + \mathcal{O}(\epsilon^3)$$

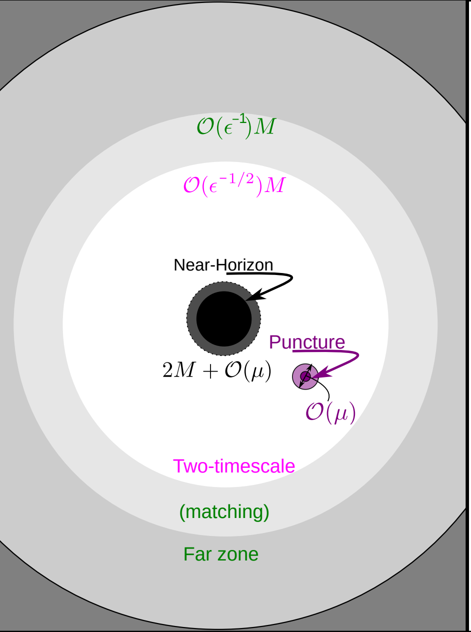
- Worldline ansatz:

$$z^\mu(t) = z^{(0)}(\tilde{t}, q^A) + \epsilon z^{(1)}(\tilde{t}, q^A) + \mathcal{O}(\epsilon^2)$$

- Precision of approximation preserved - dephasing time is $\sim$the entire inspiral, rather than the standard result for black hole perturbation theory : geometric mean of timescales
- Our method applies the two timescale approximation to metric perturbations to preserve field precision for full inspiral

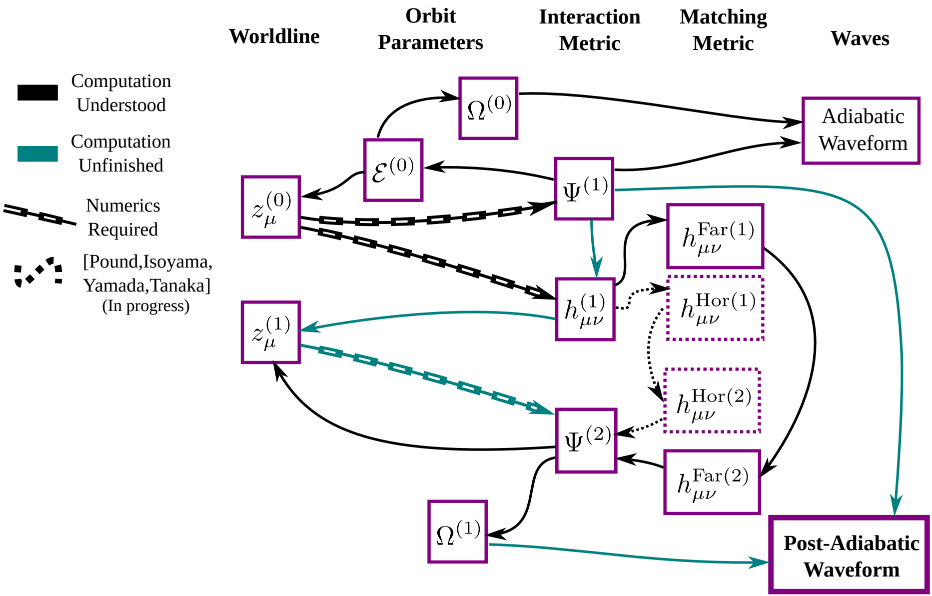


# Moving Beyond Adiabatic Precision




- **Near-Horizon** : Black hole perturbation theory [Pound, Yamada, Isoyama, Tanaka - in progress]
- Interaction zone : Two timescale expansion, worldline two-time
  - Post-adiabatic evolution requires matching to adjacent regions
- Near small object : Puncture [Pound, 2014]
- Far zone : Post-Minkowski approximation [Extending Pound 2015]

# Suggested Algorithm for Post-Adiabatic Computation



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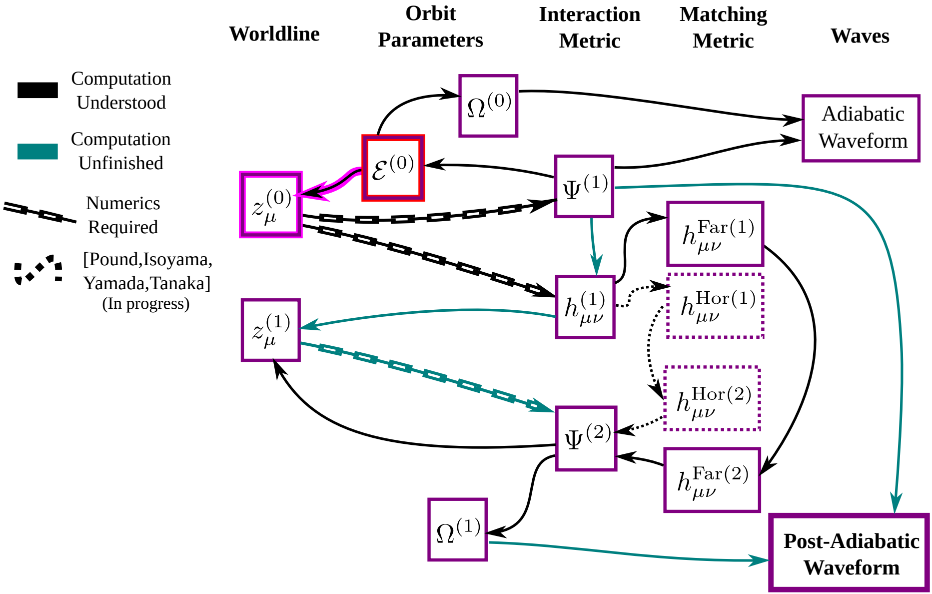

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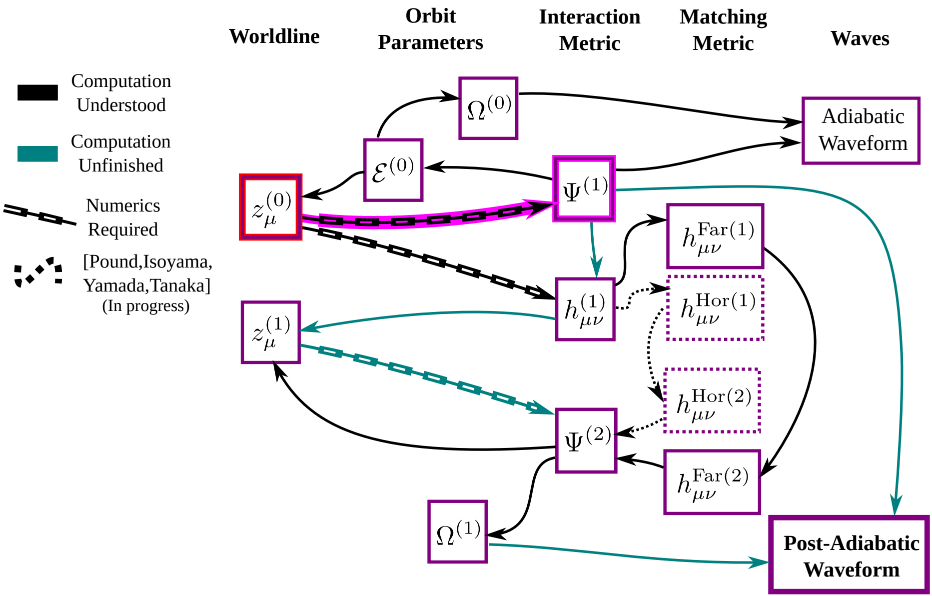

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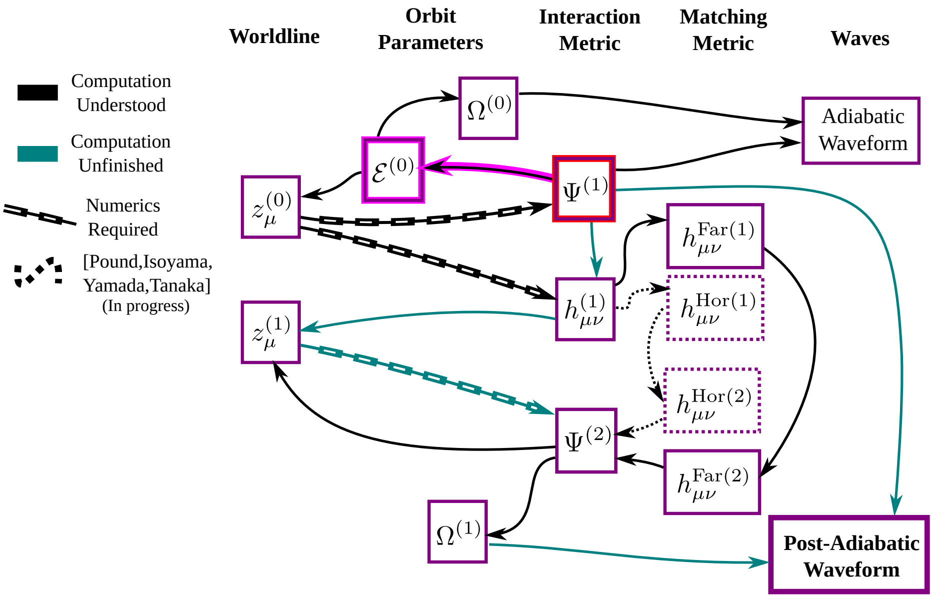

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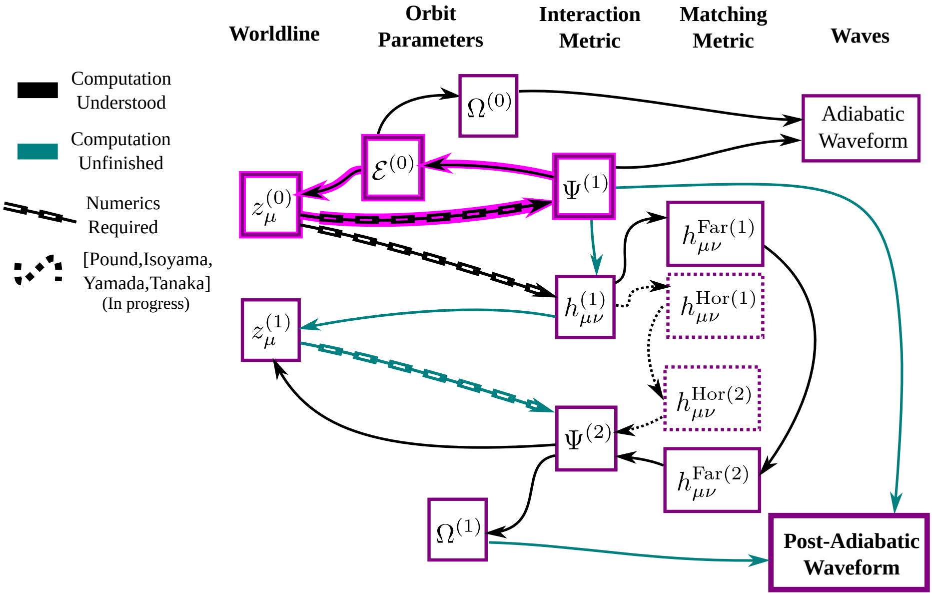

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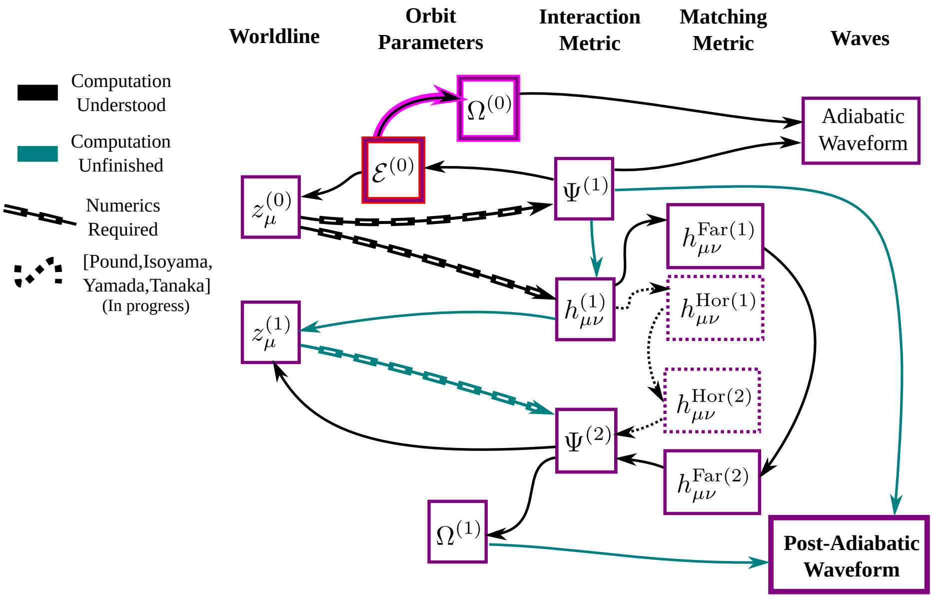

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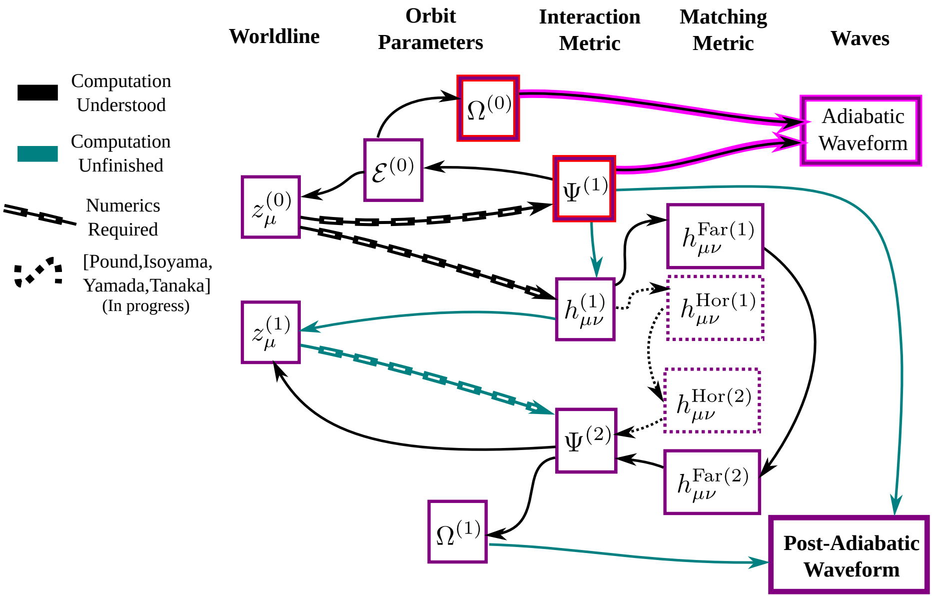

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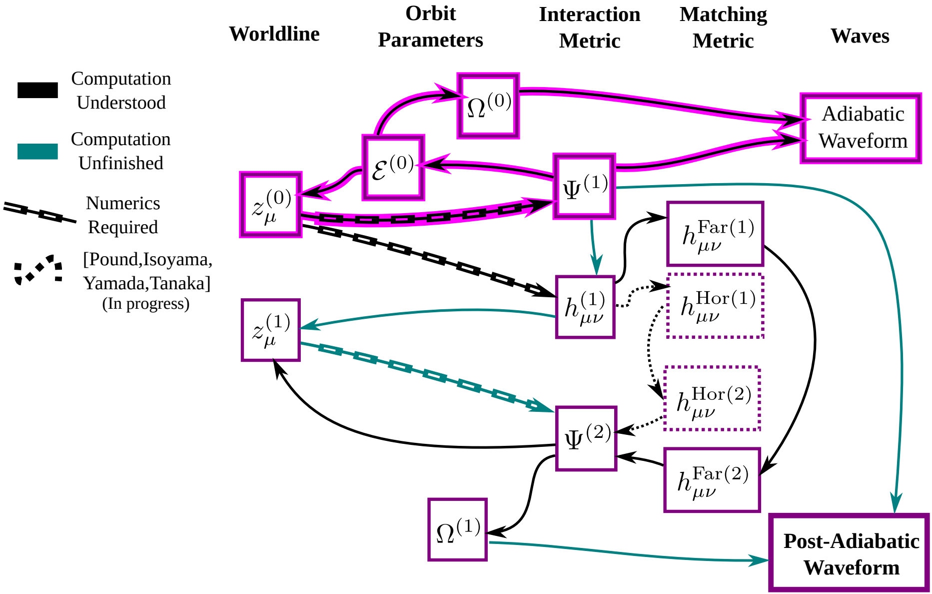

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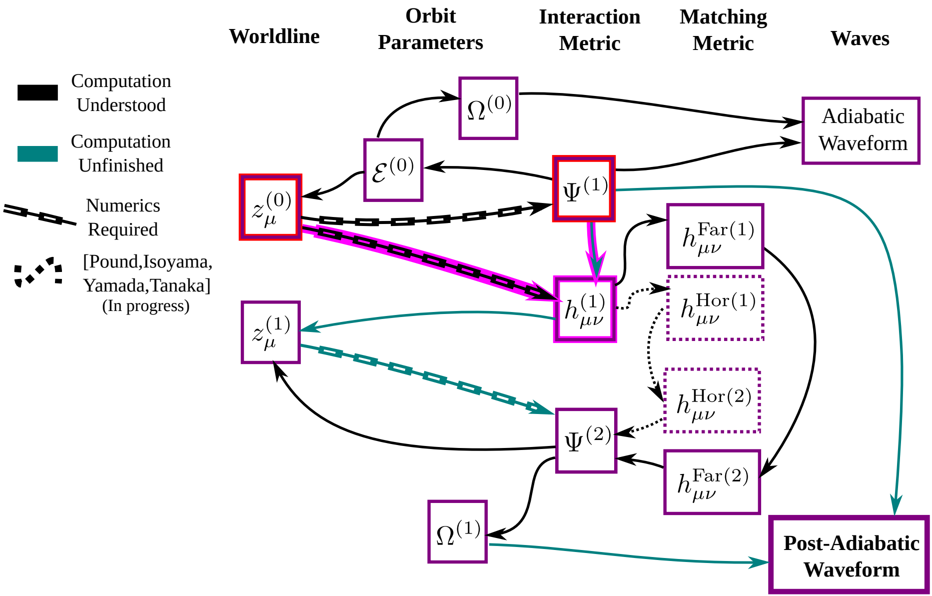

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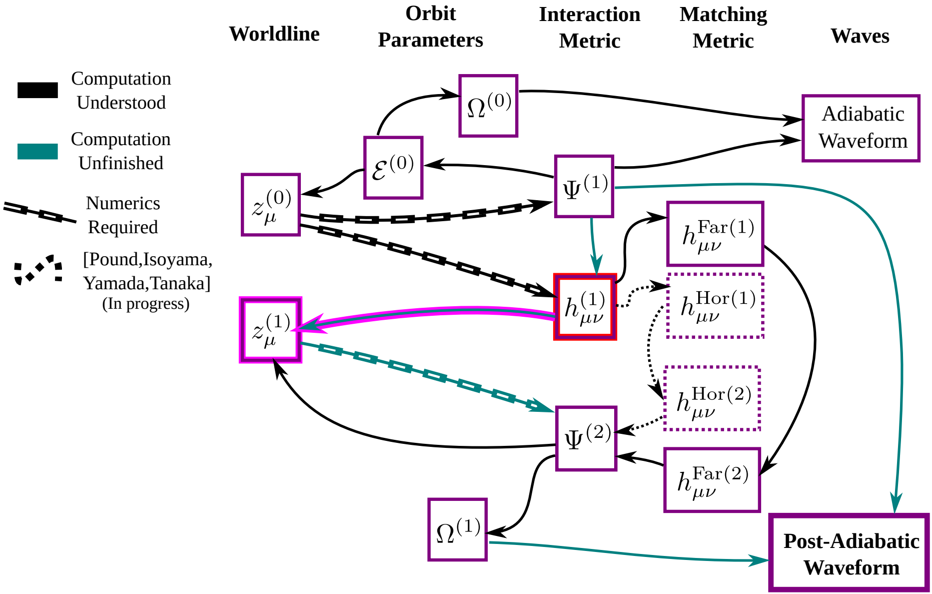

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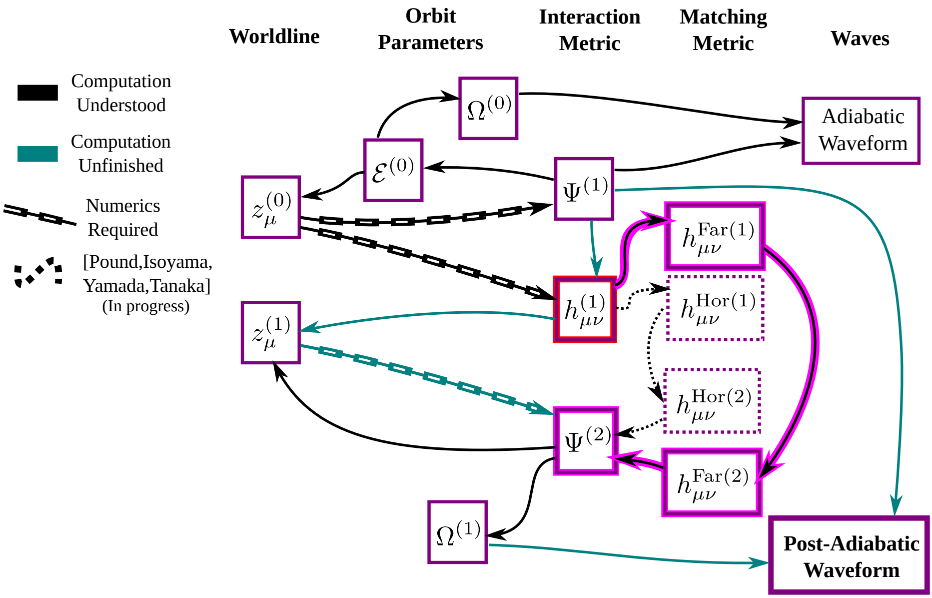

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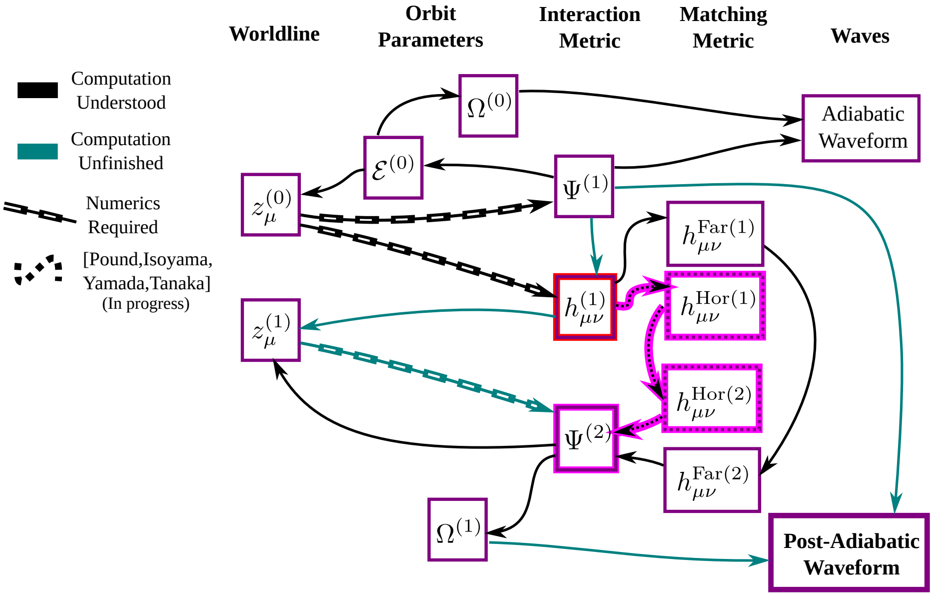

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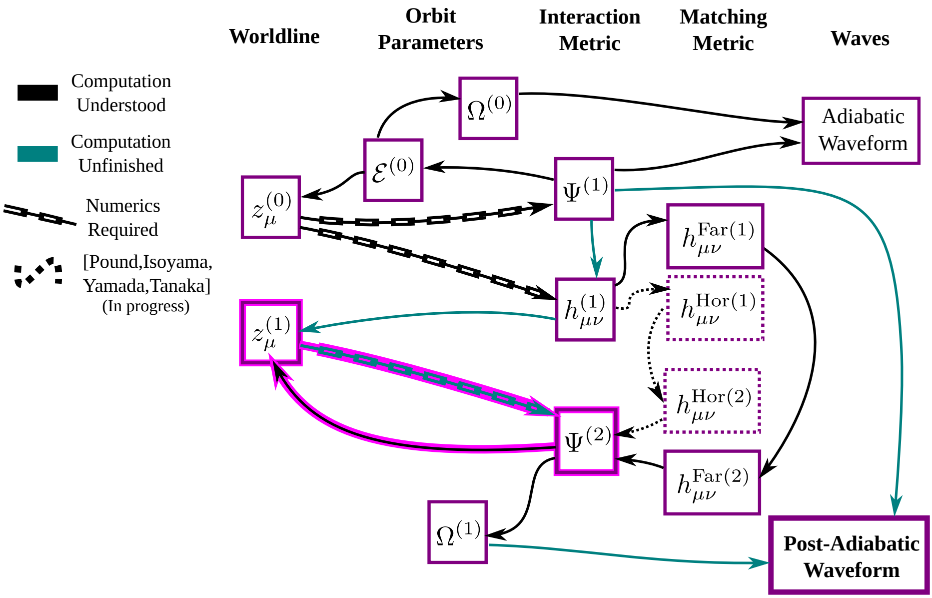

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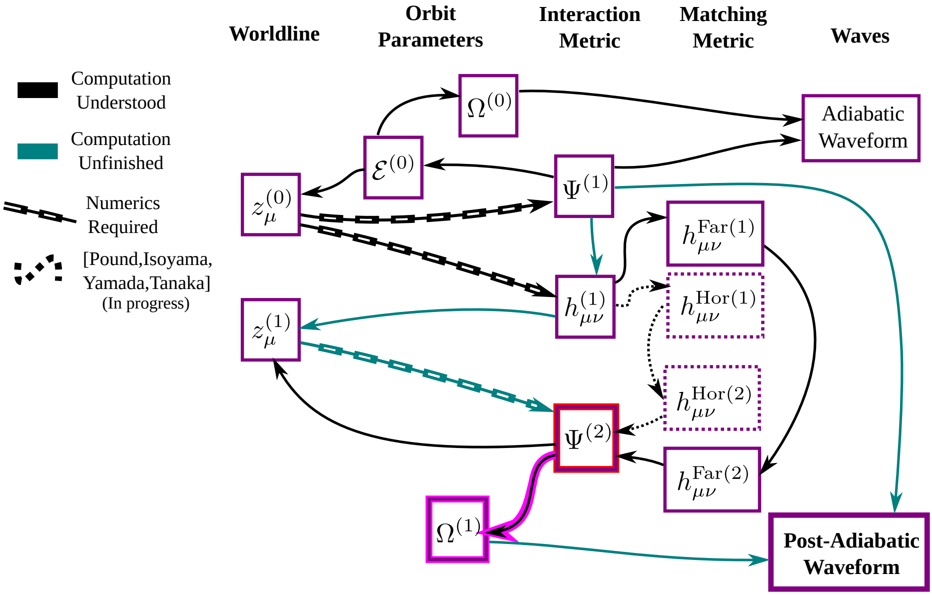

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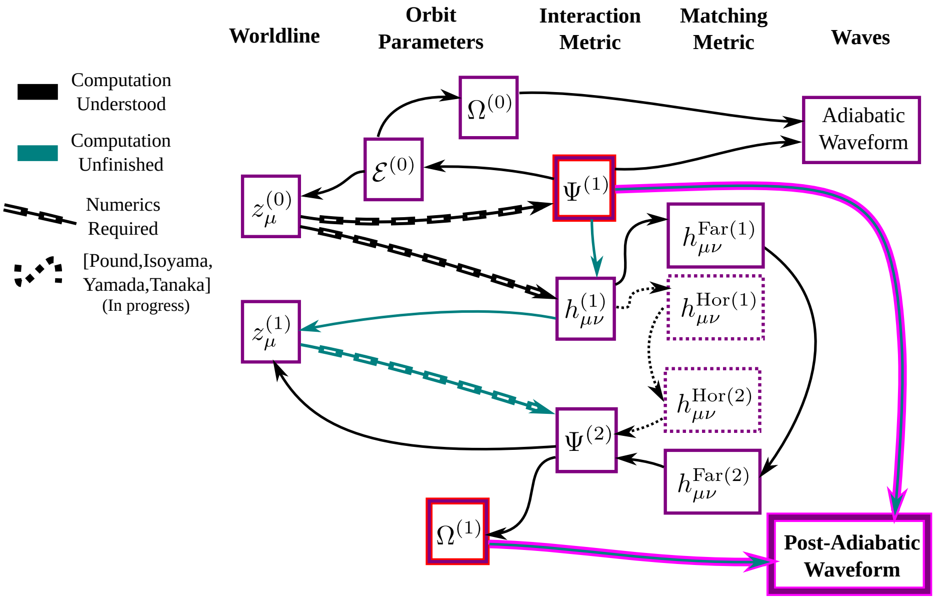

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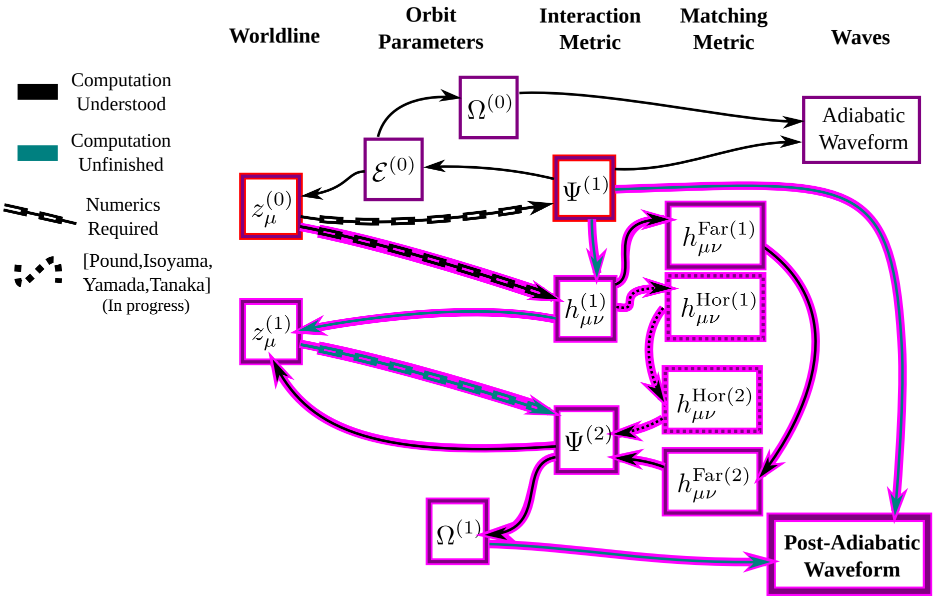
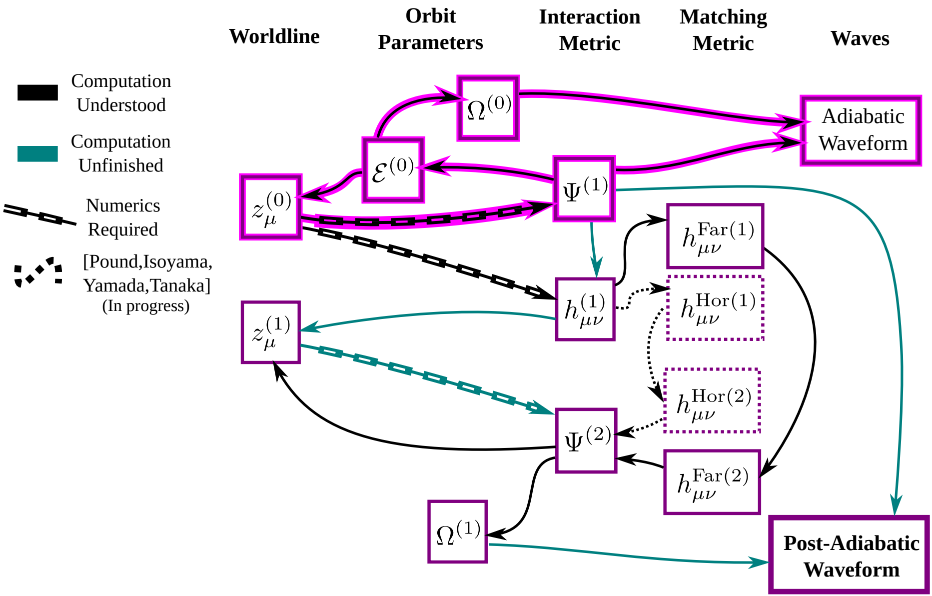
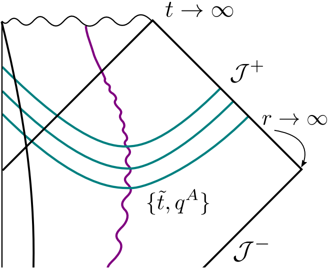



Worldline
Orbit Parameters
Interaction Metric
Matching Metric
Waves
Computation Understood
Computation Unfinished
Numerics Required
[Pound,Isoyama, Yamada,Tanaka] (In progress)
$\Omega^{(0)}$
Adiabatic Waveform
$\mathcal{E}^{(0)}$
$\Psi^{(1)}$
$z_\mu^{(0)}$
$h_{\mu\nu}^{\text{Far}(1)}$
$h_{\mu\nu}^{(1)}$
$h_{\mu\nu}^{\text{Hor}(1)}$
$z_\mu^{(1)}$
$h_{\mu\nu}^{\text{Hor}(2)}$
$\Psi^{(2)}$
$h_{\mu\nu}^{\text{Far}(2)}$
$\Omega^{(1)}$
Post-Adiabatic Waveform

# Adiabatic Order : Computational Steps




- Coupled self-consistent equations, promoted to two-timescale

$$\nabla^\alpha \nabla_\alpha h^{(1)}_{\mu\nu} + 2R_\mu{}^\alpha{}_\nu{}^\beta h^{(1)}_{\alpha\beta} = -16\pi \bar{T}_{\mu\nu}(x; z)$$

**[Self force equations of motion]**

- time derivatives replaced as:

$$\partial_t \to \epsilon \partial_{\tilde{t}} + \Omega^A(\tilde{t}, \epsilon)\partial_{q^A}$$

- Form of Adiabatic order metric perturbation:

$$h^{(1)}_{\alpha\beta} = \frac{\partial g_{\alpha\beta}}{\partial M}\delta M(\tilde{t}) + \frac{\partial g_{\alpha\beta}}{\partial a}\delta a(\tilde{t}) + \bar{F}_{\alpha\beta}(\tilde{t}, \bar{x}^i) + \tilde{F}_{\alpha\beta}(\tilde{t}, \bar{x}^i, q^A)$$

# Worldline Solutions [Hinderer,Flanagan]

- Solutions for the worldline at adiabatic order:

$$\left\langle \frac{dJ_M^{(0)}}{d\tilde{t}} \right\rangle = \left\langle G_M^{(1)} \left[ J_M^{(0)}(\tilde{t}) \right] \right\rangle$$

$$\frac{dq^{(0)A}}{dt} = \Omega^A(J^M(\tilde{t}))$$

- Forcing functions are derived from the gravitational self-force (MiSaTaQuWa at this order)
- The leading order wordline $z^{(0)\mu}$ requires only time averaged self-force $\langle G^{(1)M} \rangle$
- Angle variable equation defines the relation between fast time variables $q^A$ and physical time
  - Allows projection to a physical time dependence from fast time harmonic expansion

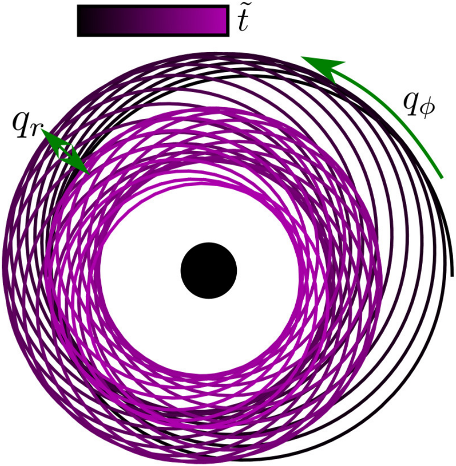
$$f(t) = \sum_{k_A} a_{k_A}(\tilde{t}) e^{iq_A k^A}$$

# Adiabatic Order Computation

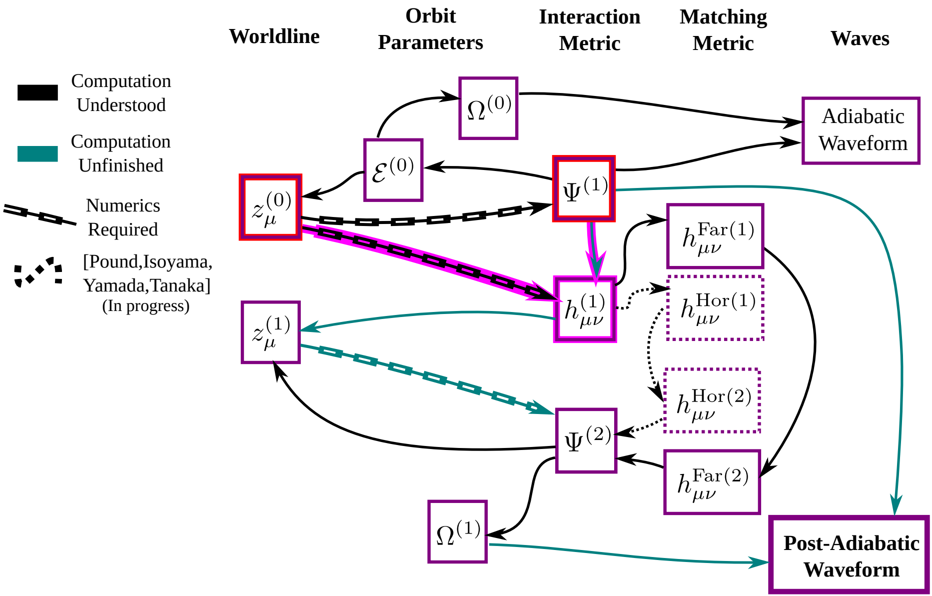
- Leading order worldline - biperiodic geodesic at fixed $\tilde{t}$: $z^{(0)}[E^{(0)}(\tilde{t}), L_z^{(0)}(\tilde{t}), q^A]$
- Similar at this order to osculating geodesics [Pound, Poisson 2007]
- Procedure outline:
  - Evaluate $\Psi_{RW/Z}(\tilde{t})$ harmonics
  - Use fluxes to determine first-order averaged self-force and worldline

$$\left\langle \frac{d\mathcal{E}}{d\tau} \right\rangle = \langle \textbf{[Fluxes]} \rangle$$

  - Use frequencies $\Omega^{(0)A}(\tilde{t})$ to project $\{\tilde{t}, q^A\}$ to physical time



# Post-Adiabatic : Interaction Zone



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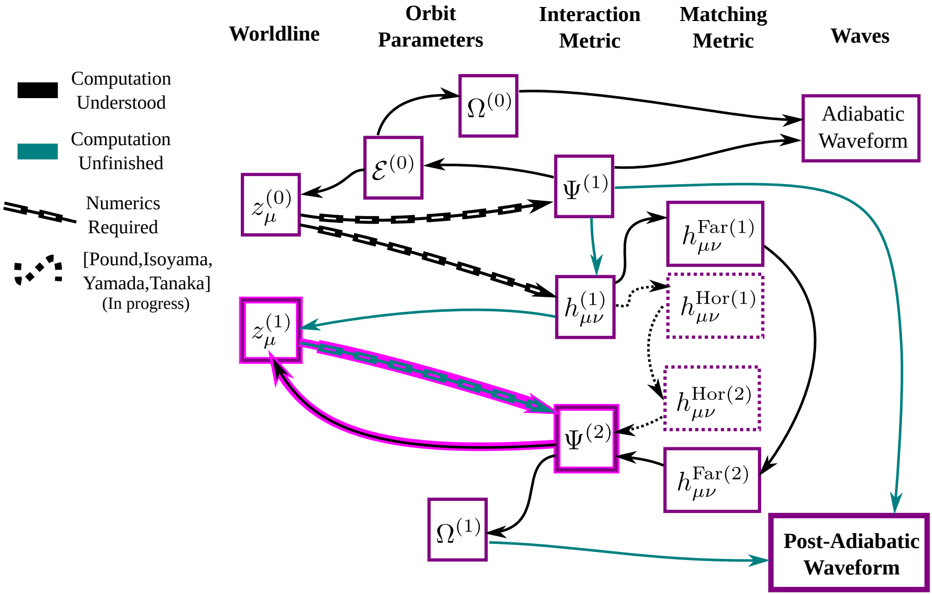

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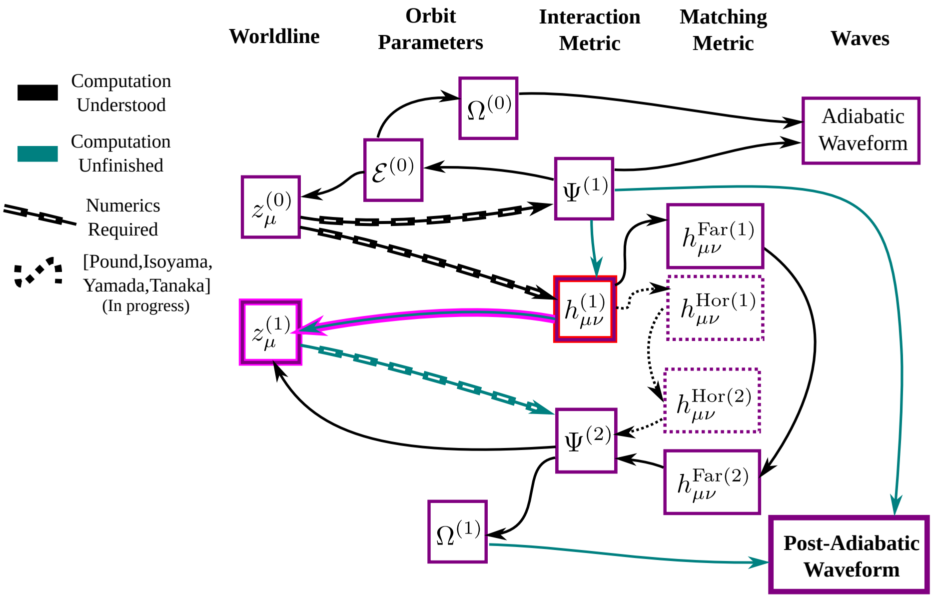

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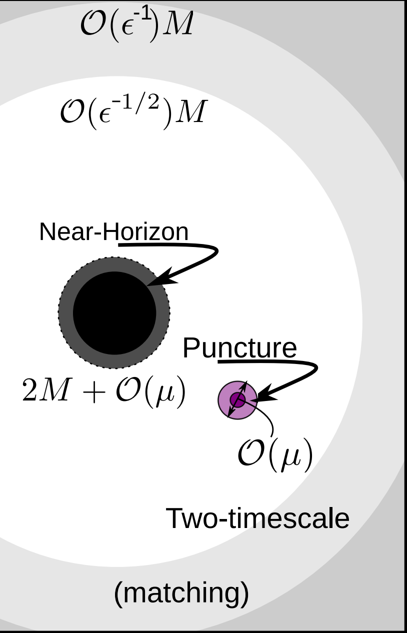



- (Second order) dissipative part - our goal is to use fluxes $\delta\Psi_{RW/Z}$ [In development]
  - quadratic source at second order
  - Need to fully understand fluxes $\leftrightarrow$ orbit evolution at second order
- (First order) conservative self-force - frequency shifts and orbit wobble
  - $L_z^{(1)}(q^A, \tilde{t})$, $E^{(1)}(q^A, \tilde{t})$
  - $dq^A/dt = \Omega^A(L_z, E) + \epsilon g^A(\tilde{t}, q^A)$
  - Requires full metric solution to first order, not just time-averaged adiabatic fluxes
- Drift of central black hole $\delta M(\tilde{t})$ and $\delta a(\tilde{t})$

# Fluxes at Second Order

- Metric perturbations expressed as a sum of a known puncture and a residual field

$$h_{\alpha\beta}^{(1)} - h_{\alpha\beta}^{(1)\mathcal{P}} = h_{\alpha\beta}^{(1)\mathcal{R}}$$

- Time-averaged orbit parameters can be derived using second order self force [Pound;Gralla] in the two-timescale orbit expressions
- Expression for time-averaged energy $E$, angular momentum $L_z$ evolution:

$$\left\langle \frac{d\mathcal{E}}{d\tau} \right\rangle = \frac{1}{4}\langle u^\gamma u^\delta \partial_\xi h_{\gamma\delta}^{(2)\mathcal{R}} \rangle + \frac{1}{8}\left\langle u^\alpha u^\beta u^\gamma u^\delta \partial_\xi ( h_{\alpha\beta}^{(1)\mathcal{R}} h_{\gamma\delta}^{(1)\mathcal{R}} ) \right\rangle$$

- Suspected to be gauge-invariant for gauges that preserve the two-timescale ansatz- need to confirm

# Near-Identity Transformation

- First order conservative SF gives fast-time wobble in orbit parameters
- Near-Identity transformation [Vines,Flanagan] moves this dependence to the worldline function

$$
\begin{aligned}
J'^M =& J^M + \epsilon T^M(J^M, q^A, \tilde{t}) \\
Q^A =& q^A + \epsilon L^A(J^M, q^A, \tilde{t})
\end{aligned}
$$

- A carefully chosen set of parameters can bring the orbit equations of motion to

$$
\begin{aligned}
\frac{dQ_A}{dt} =& \Omega_A(J'^M(\tilde{t})) + \epsilon \langle g_A^{(1)}[J'^M(\tilde{t})] \rangle + \mathcal{O}(\epsilon) \\
\frac{J'^M}{dt} =& \epsilon \langle G^{(1)M}[J'^M(\tilde{t})] \rangle
\end{aligned}
$$

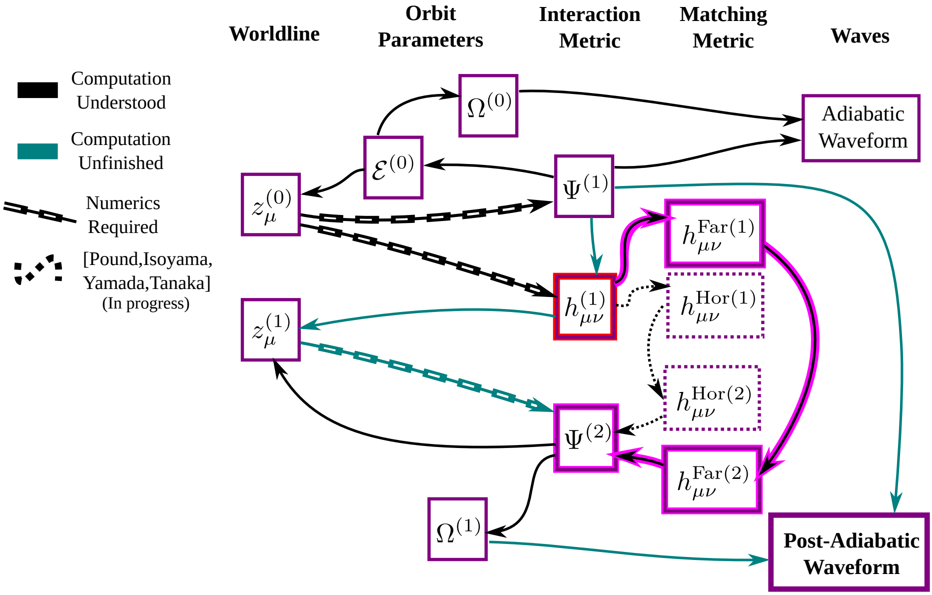
# Puncture Corrections

- From an explicit formulation of the worldline $z^{(0)\mu} + \epsilon z^{(1)\mu} + \mathcal{O}(\epsilon^2)$, require a near-particle metric
- This puncture metric enters into the near-particle relaxed EFE at second order:

$$E_{\mu\nu}[h^{(2)\mathcal{R}}_{\alpha\beta}] = -E_{\mu\nu}[h^{(2)\mathcal{P}}_{\alpha\beta}] + S_{\mu\nu}[h^{(1)}_{\alpha\beta}, h^{(1)}_{\alpha\beta}] + \delta T_{\mu\nu}$$

- Previous covariant puncture results [Pound] through second order for an exact wordline
- These results need to be adjusted slightly for the correction from a $\mathcal{O}(\mu)$ displacement in worldline position
  - Dipole-type corrections to the puncture, even for a spherically symmetric object
  - Details yet to be worked out in full generality - Adam has done this for quasicircular orbits, generalizable procedure given in [Pound 2015]

# Post-Adiabatic : Far and Very Far Zone



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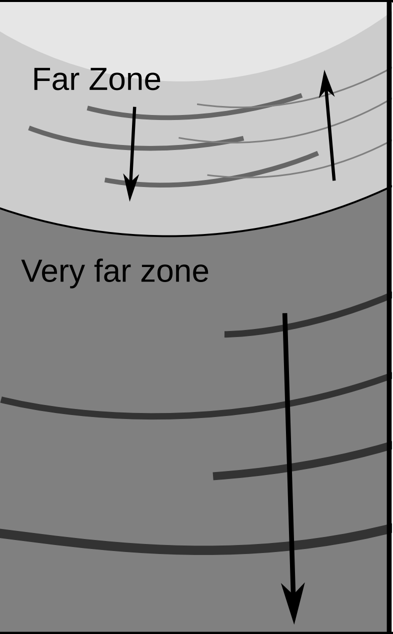


- Incoming homogeneous modes from quadratic source outside interaction zone $r > M/\epsilon$ [scalar analogue - Pound 2015]
  - Outgoing modes give $\mathcal{O}(\epsilon^2)$ quadratic source
  - Post-Minkowski expansion in far zone
  - Post-Minkowski techniques developed by [Blanchet and Damour; Will et al]
  - Primarily m $= 0$, few components of tensor harmonics for most geodesics
- History-dependent outgoing radiation gets secular enhancement at $r \gg 1/\epsilon$
  - $\mathcal{O}(\epsilon^2)$ flux from far zone build to $\mathcal{O}(\epsilon)$ in very far

# Matching with the Far Zone

- Two-time formalism does not keep the desired precision at $r \gg M$ at second order, even for hyperboloidal slices
- Information from the far zone required to completely determine the subleading worldline
- Lorenz gauge, so we seek solutions to

$$\Box h^{(1)\mathsf{Far}}_{\mu\nu} = 0$$
$$\Box h^{(2)\mathsf{Far}}_{\mu\nu} = S_{\mu\nu}$$

- $h^{(1)\mathsf{Far}}_{\mu\nu}$ obtained from interaction zone information
- Both quasicircular and elliptical orbits' $S_{\mu\nu}$:
  - Only static component is required - ensures a direct sum of modes in no-resonance assumption
  - Source depends only on traceless angular metric components ($\propto Y_{AB}; \propto X_{AB}$)

# Far Zone Source

- Static far zone source, generalized to elliptical orbits (quasicircular retrieved by discarding $n \neq 0$ radial harmonics)

$$\begin{aligned} s_{tt}^{\bar{l}0} =& s_{rr}^{\bar{l}0} = -s_{tr}^{\bar{l}0} \\ =& \sum_{l,l'>2} \sum_{m\neq 0} \sum_{n>0} (n\Omega_r + m\Omega_\phi)^2 \\ & * \left(C_{lm-2\,l'-m,2}^{l00} + C_{lm2\,l'-m,-2}^{l00}\right) \\ & * \left(h_{l'm}^{(7)} h_{lm}^{(7)} + h_{l'm}^{(10)} h_{lm}^{(10)}\right) \end{aligned}$$

- (7),(10) are [Barack,Sago] traceless angular components
  - Correspond to linear combinations of coefficients of $Y_{AB}$, $X_{AB}$ in [Martel,Poisson] decomposition

# Far Zone Inhomogeneous Solution

- Results of applying [Blanchet, Damour] to find finite part of $\Box_{\text{ret}}^{-1}(r^{-2} s_{sL}^{\bar{l}\bar{n}} \hat{n}^L)$
- Combining results for radial and angular periodicities (in an STF decomposition; no-resonance assumption $\Rightarrow \bar{m} = 0$):

$$j_{sL}^{\bar{l}\bar{n}} = \begin{cases} s_{sL}^{00}(\bar{w})\left(\ln(\frac{2r}{\epsilon}) - 1\right) - \int_0^\infty d\bar{z} \dot{\bar{s}}_{sL}^{00}(\bar{w} - \bar{z}) \ln \bar{z} & \bar{n} = 0;\ \bar{l} = 0 \\ -\frac{\hat{s}_{sL}^{\bar{l}0}}{\bar{l}(\bar{l}+1)} & \bar{n} = 0;\ \bar{l} > 0 \\ \frac{-i s_{sL}^{\bar{l}\bar{n}} \ln(r)}{2r\bar{n}\Omega_r} & \bar{n} > 0 \end{cases}$$

- Written in terms of a scaled retarded time variable $\bar{w}, \bar{z}$
- These expressions, combined with the sources above represent the full set of information required from the Far zone to compute the perturbations to the near-zone worldline

# Summary and Conclusions

- Two-timescale together with matched expansions provide a framework for handling the subtle secular evolution of inspirals
- Adiabatic order largely a reformulation of existing derivations
- Post-adiabatic is a work in progress, but promises important improvements - $\mathcal{O}(\epsilon)$ phase error throughout inspiral
  - Some technical details of Schwarzschild remain to be worked out
- Hoped-for future steps:
  - **Kerr!** (eventually...)
  - resonances