High Order Analytical Self-Force Calculations

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Accuracy/validity of conservative GSF PN calculations

Akcay et al 2012- h_{uu} has a pole at $r = 3M$.

Bini et al 2016- radius of convergence at best $\frac{1}{3}$

$$c_n \sim 3^{-n}$$

if we want x digits at $k-$PN

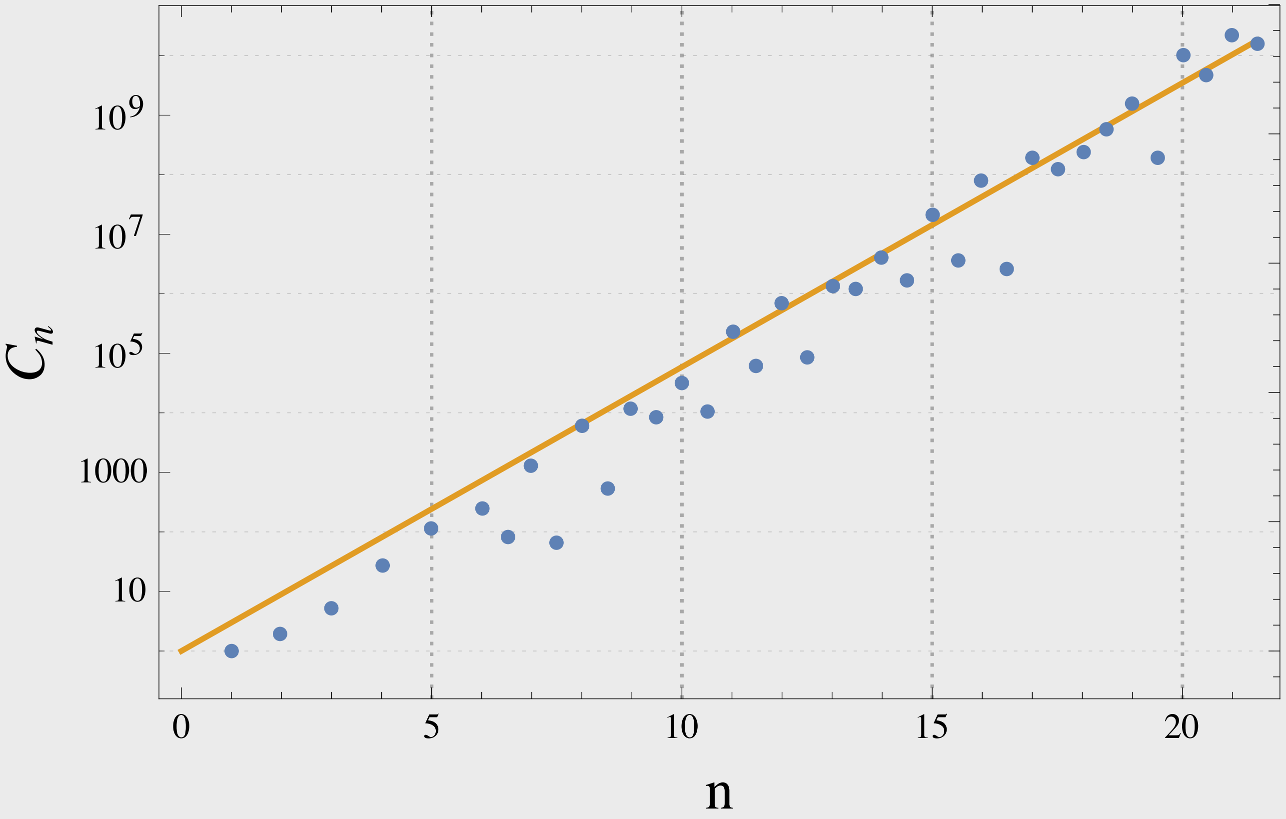
$$\frac{3^{k+1}}{r_p^{k+1}} < 10^{-x}$$

then

$$k > \frac{x}{\log_{10} \frac{r_p}{3}} - 1$$

10PN- 16 digits at $r = 70M$

20PN- 16 digits at $r = 15M$



Analytics with MST series- large radius, small frequency

$$\left.\begin{array}{l} {}_sR^{\rm up}_{\ell m\omega}(r) \\ {}_sR^{\rm in}_{\ell m\omega}(r) \end{array}\right. \rightarrow \text{Weyl scalars } \psi_4, \psi_0 \quad \text{at } r = r_p \text{ large}$$

$$\omega = m\Omega_\varphi, m\Omega_\varphi + n\Omega_\theta, m\Omega_\varphi + k\Omega_r$$

$$\approx \mathcal{O}(r_p^{-3/2})$$

$$\left\{ \begin{array}{l} X_1 = \frac{GM}{r} \\ X_2 = \omega^2 r^2 \end{array} \right.$$

expand rather in $\frac{1}{c}$

Units $\frac{1}{c^2}$

$\rightarrow$ expansions of the homogeneous solutions in both r and ω

Analytics with MST series- large radius, small frequency

$$ {}_sR^{\rm in}_{\ell m\omega}(r) = C^{\nu}_{(\rm in)}(x)\sum_{n=-\infty}^{\infty} a^{\nu}_{n}\,{}_2F_1(a,b,c,x) \qquad {}_sR^{\rm up}_{\ell m\omega}(r) = C^{\nu}_{(\rm up)}(x)\sum_{n=-\infty}^{\infty} a^{\nu}_{n}(2iz)^n U(a,b,-2iz) $$

$$ \alpha^{\nu}_{n} a_{n+1} + \beta^{\nu}_{n} a_{n} + \gamma^{\nu}_{n} a_{n-1} = 0 $$

$$ \left.\begin{matrix}\alpha^{\nu}_{n}\\ \beta^{\nu}_{n}\\ \gamma^{\nu}_{n}\end{matrix}\right\} q, s, \omega, \ell, m, \nu, {}_s\lambda_{\ell m} $$

$$ \begin{matrix} R_n = \frac{a^{\nu}_{n}}{a^{\nu}_{n-1}} \\ L_n = \frac{a^{\nu}_{n}}{a^{\nu}_{n+1}} \end{matrix} \rightarrow \begin{matrix} R_n = -\frac{\gamma^{\nu}_{n}}{\beta^{\nu}_{n} + \alpha^{\nu}_{n} R_{n+1}} \\ L_n = -\frac{\alpha^{\nu}_{n}}{\beta^{\nu}_{n} + \gamma^{\nu}_{n} L_{n-1}} \end{matrix} \qquad a_n = \begin{cases} R_n a_{n-1} & : n > 0 \\ L_n a_{n+1} & : n < 0 \end{cases} $$

For consistency, ν is picked so that $R_n(\nu)L_{n-1}(\nu) = 1.$

This is all perfect for numerics- one of the faster parts of the calculation.

Analytics with MST series- large radius, small frequency

$$R_n = -\frac{\gamma_n^\nu}{\beta_n^\nu + \alpha_n^\nu R_{n+1}}$$
$$L_n = -\frac{\alpha_n^\nu}{\beta_n^\nu + \gamma_n^\nu L_{n-1}}$$

$\rightarrow$ expanding $\alpha_n^\nu, \beta_n^\nu, \gamma_n^\nu$ for low ω gives a structure for changing n

$$a_n = \begin{cases} R_n a_{n-1} & : n > 0 \\ L_n a_{n+1} & : n < 0 \end{cases}$$

$l = 2$

a_5	0	0	0	0	0	ϵ^5		
a_4	0	0	0	0	ϵ^4			
a_3	0	0	0	ϵ^3				
a_2	0	0	ϵ^2					
a_1	0	ϵ						
a_0	1							
a_{-1}	0	0	0	ϵ^3				
a_{-2}	0	0	0	0	ϵ^4			
a_{-3}	0	0	0	0	ϵ^4			
a_{-4}	0	0	0	0	0	ϵ^5		
a_{-5}	0	0	0	0	ϵ^4			
a_{-6}	0	0	0	0	0	ϵ^5		
a_{-7}	0	0	0	0	0	0	ϵ^6	
a_{-8}	0	0	0	0	0	0	0	ϵ^7

$l = 3$

a_5	0	0	0	0	0	ϵ^5		
a_4	0	0	0	0	ϵ^4			
a_3	0	0	0	ϵ^3				
a_2	0	0	ϵ^2					
a_1	0	ϵ						
a_0	1							
a_{-1}	0	ϵ						
a_{-2}	0	0	0	0	ϵ^4			
a_{-3}	0	0	0	0	0	ϵ^5		
a_{-4}	0	0	0	0	0	ϵ^5		
a_{-5}	0	0	0	0	0	0	ϵ^6	
a_{-6}	0	0	0	0	0	0	0	ϵ^7
a_{-7}	0	0	0	0	0	0	ϵ^6	
a_{-8}	0	0	0	0	0	0	0	ϵ^7

$l = 5$

a_5	0	0	0	0	0	ϵ^5				
a_4	0	0	0	0	ϵ^4					
a_3	0	0	0	ϵ^3						
a_2	0	0	ϵ^2							
a_1	0	ϵ								
a_0	1									
a_{-1}	0	ϵ								
a_{-2}	0	0	ϵ^2							
a_{-3}	0	0	0	ϵ^3						
a_{-4}	0	0	0	0	0	ϵ^6				
a_{-5}	0	0	0	0	0	0	0	ϵ^7		
a_{-6}	0	0	0	0	0	0	0	ϵ^7		
a_{-7}	0	0	0	0	0	0	0	0	ϵ^8	
a_{-8}	0	0	0	0	0	0	0	0	0	ϵ^9

Analytics with MST series- large radius, small frequency

$$g_n(\omega) = \alpha_n^{\nu} a_{n+1}^{\nu} + \beta_n^{\nu} a_n^{\nu} + \gamma_n^{\nu} a_{n-1}^{\nu}$$

$$a_n^{\nu} = \sum_{i=i_{\min}}^{\infty} a_n^{\nu,(i)} \omega^i \qquad \nu = \ell + \nu^{(2)}(a\omega)^2 + \nu^{(3)}(a\omega)^3 + \ldots$$

$l = 2$

a_5	0	0	0	0	0	ϵ^5		
a_4	0	0	0	0	ϵ^4			
a_3	0	0	0	ϵ^3				
a_2	0	0	ϵ^2					
a_1	0	ϵ						
a_0	1							
a_{-1}	0	0	0	ϵ^3				
a_{-2}	0	0	0	0	ϵ^4			
a_{-3}	0	0	0	0	ϵ^4			
a_{-4}	0	0	0	0	0	ϵ^5		
a_{-5}	0	0	0	0	ϵ^4			
a_{-6}	0	0	0	0	0	ϵ^5		
a_{-7}	0	0	0	0	0	0	ϵ^6	
a_{-8}	0	0	0	0	0	0	0	ϵ^7

$l = 3$

a_5	0	0	0	0	0	ϵ^5		
a_4	0	0	0	0	ϵ^4			
a_3	0	0	0	ϵ^3				
a_2	0	0	ϵ^2					
a_1	0	ϵ						
a_0	1							
a_{-1}	0	ϵ						
a_{-2}	0	0	0	0	ϵ^4			
a_{-3}	0	0	0	0	0	ϵ^5		
a_{-4}	0	0	0	0	0	ϵ^5		
a_{-5}	0	0	0	0	0	0	ϵ^6	
a_{-6}	0	0	0	0	0	0	0	ϵ^7
a_{-7}	0	0	0	0	0	0	ϵ^6	
a_{-8}	0	0	0	0	0	0	0	ϵ^7

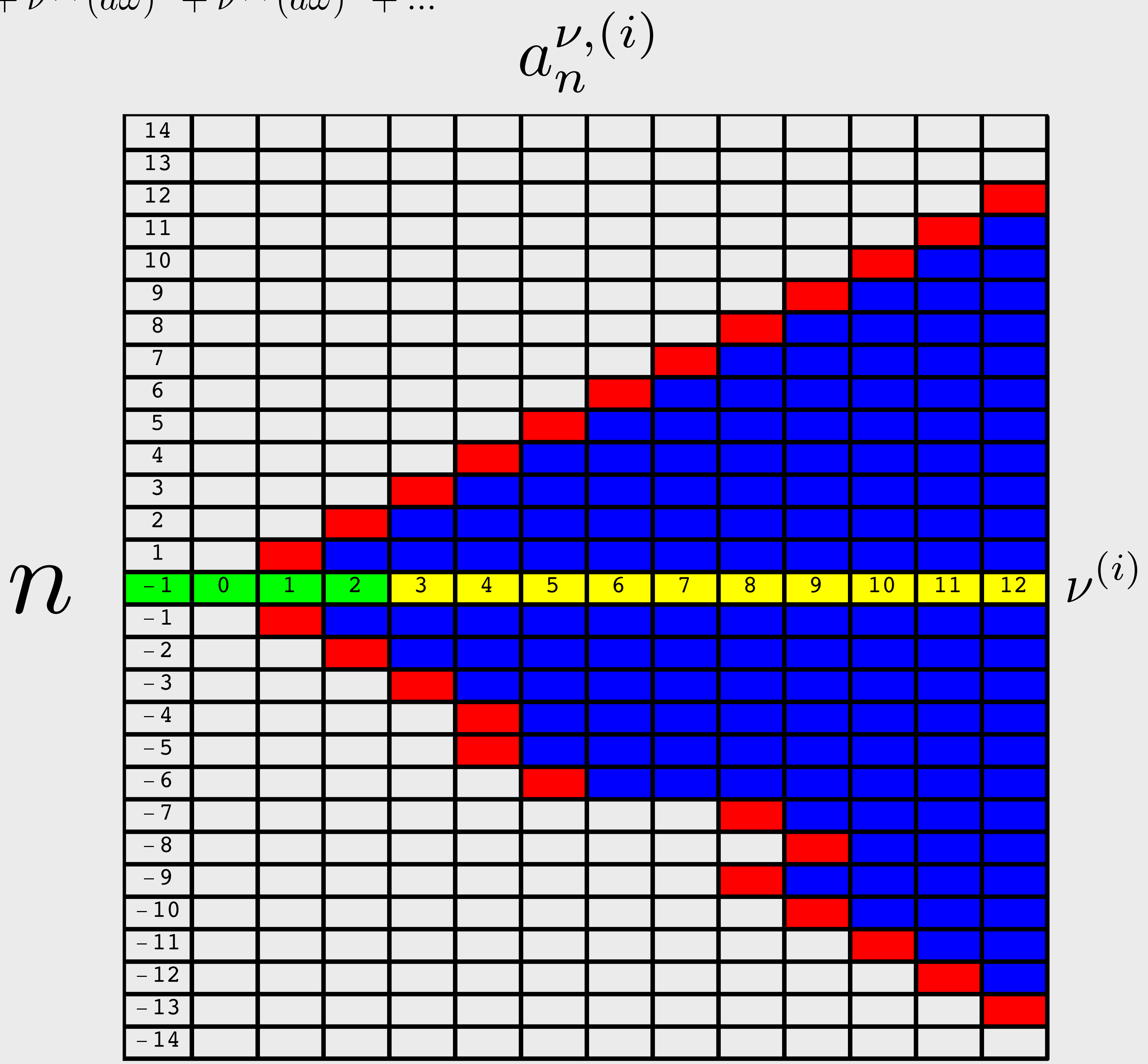
$l = 5$

a_5	0	0	0	0	0	ϵ^5				
a_4	0	0	0	0	ϵ^4					
a_3	0	0	0	ϵ^3						
a_2	0	0	ϵ^2							
a_1	0	ϵ								
a_0	1									
a_{-1}	0	ϵ								
a_{-2}	0	0	ϵ^2							
a_{-3}	0	0	0	ϵ^3						
a_{-4}	0	0	0	0	0	ϵ^6				
a_{-5}	0	0	0	0	0	0	0	ϵ^7		
a_{-6}	0	0	0	0	0	0	0	ϵ^7		
a_{-7}	0	0	0	0	0	0	0	0	ϵ^8	
a_{-8}	0	0	0	0	0	0	0	0	0	ϵ^9

Analytics with MST series- large radius, small frequency

$$g_n(\omega) = \alpha_n^\nu a_{n+1}^\nu + \beta_n^\nu a_n^\nu + \gamma_n^\nu a_{n-1}^\nu$$

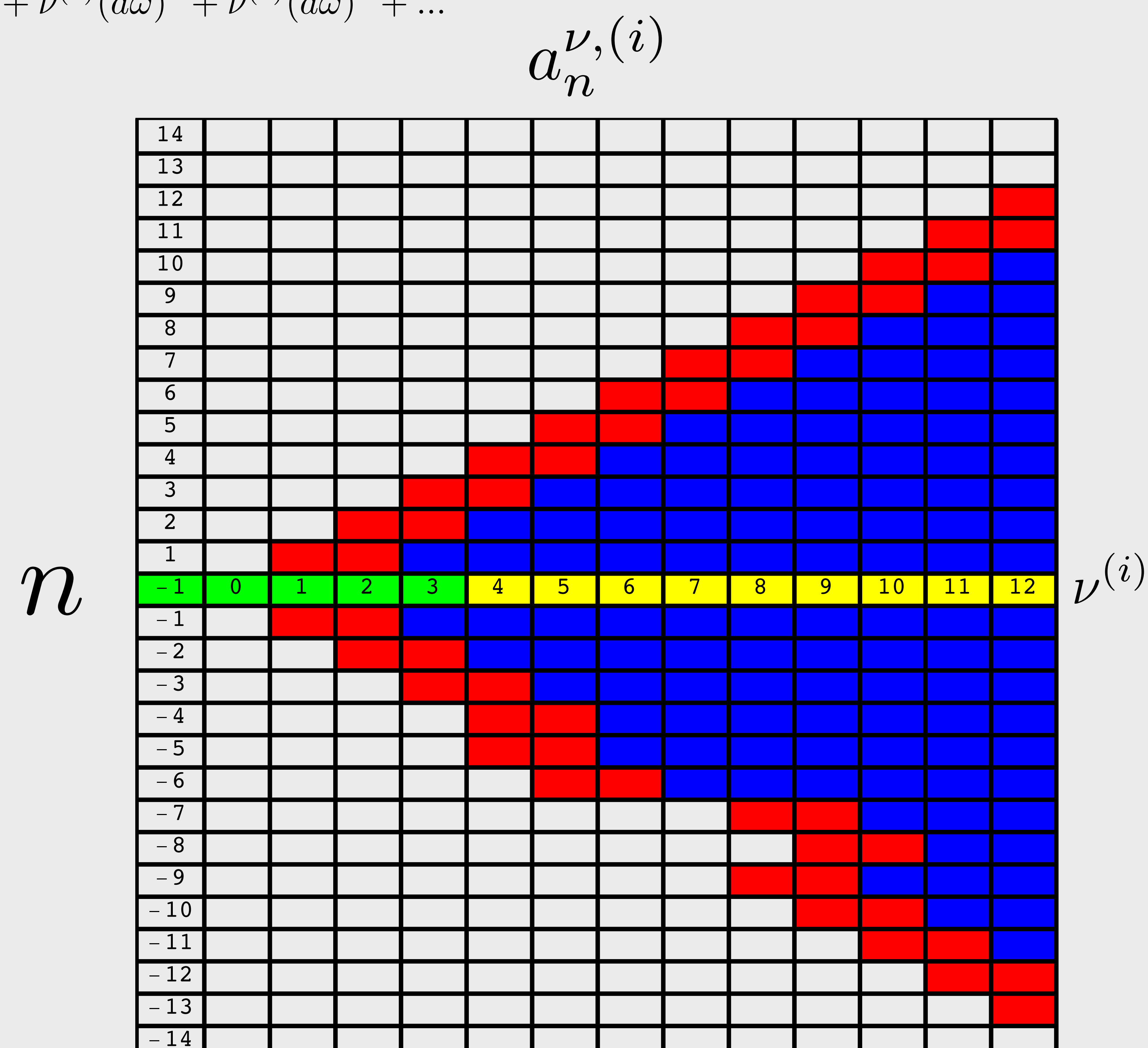
$$a_n^\nu = \sum_{i=i_{\min}}^{\infty} a_n^{\nu,(i)} \omega^i \qquad \nu = \ell + \nu^{(2)}(a\omega)^2 + \nu^{(3)}(a\omega)^3 + \ldots$$



Analytics with MST series- large radius, small frequency

$$g_n(\omega) = \alpha_n^\nu a_{n+1}^\nu + \beta_n^\nu a_n^\nu + \gamma_n^\nu a_{n-1}^\nu$$

$$a_n^\nu = \sum_{i=i_{\min}}^{\infty} a_n^{\nu,(i)} \omega^i \qquad \nu = \ell + \nu^{(2)}(a\omega)^2 + \nu^{(3)}(a\omega)^3 + \ldots$$



Analytics with MST series- large radius, small frequency

$$g_n(\omega) = \alpha_n^\nu a_{n+1}^\nu + \beta_n^\nu a_n^\nu + \gamma_n^\nu a_{n-1}^\nu$$

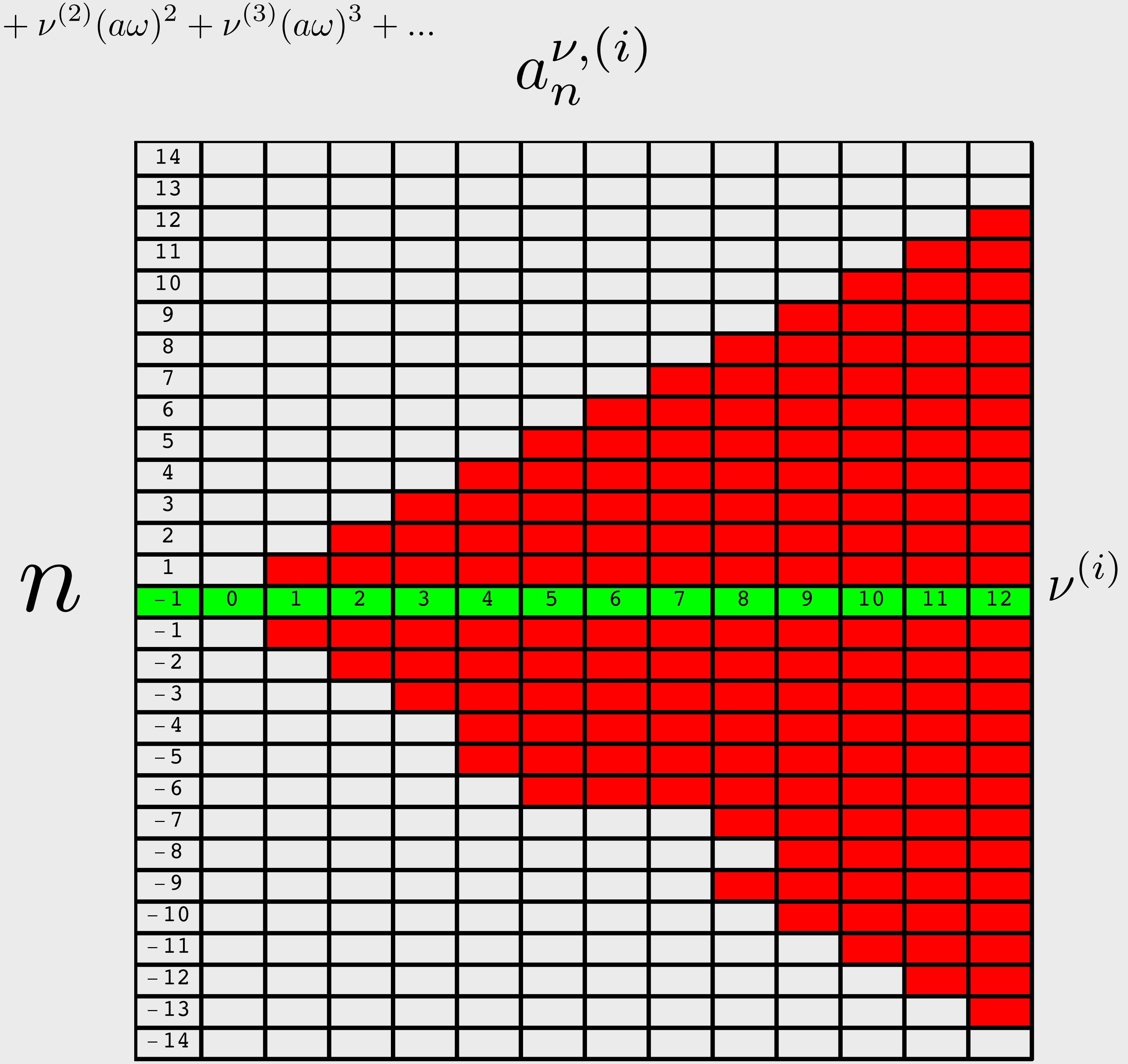
$$a_n^\nu = \sum_{i=i_{\min}}^{\infty} a_n^{\nu,(i)} \omega^i \qquad \nu = \ell + \nu^{(2)}(a\omega)^2 + \nu^{(3)}(a\omega)^3 + \ldots$$

48 hours

$n = 102,$

$a_n^\nu, \nu \sim \mathcal{O}(\epsilon^{102})$

$\to$ 50PN



Analytics with MST series- large radius, small frequency

With the coefficients in hand proceeding with calculating the Weyl scalar is systematic

$${}_sR^{\text{in}}_{\ell m\omega}(r) = C^{\nu}_{(\text{in})}(x) \sum_{n=-\infty}^{\infty} a^{\nu}_n {}_2F_1(a,b,c,x) \qquad {}_sR^{\text{up}}_{\ell m\omega}(r) = C^{\nu}_{(\text{up})}(x) \sum_{n=-\infty}^{\infty} a^{\nu}_n (2iz)^n U(a,b,-2iz)$$

Quite messy for low l, keeping everything exact in a introduces more special functions

$${}_sR^{\text{in}}_{\ell m\omega} = N(\omega)(1 + \frac{b_1}{c} + \frac{b_2}{c^2} + \frac{b_3}{c^3} + \ldots$$

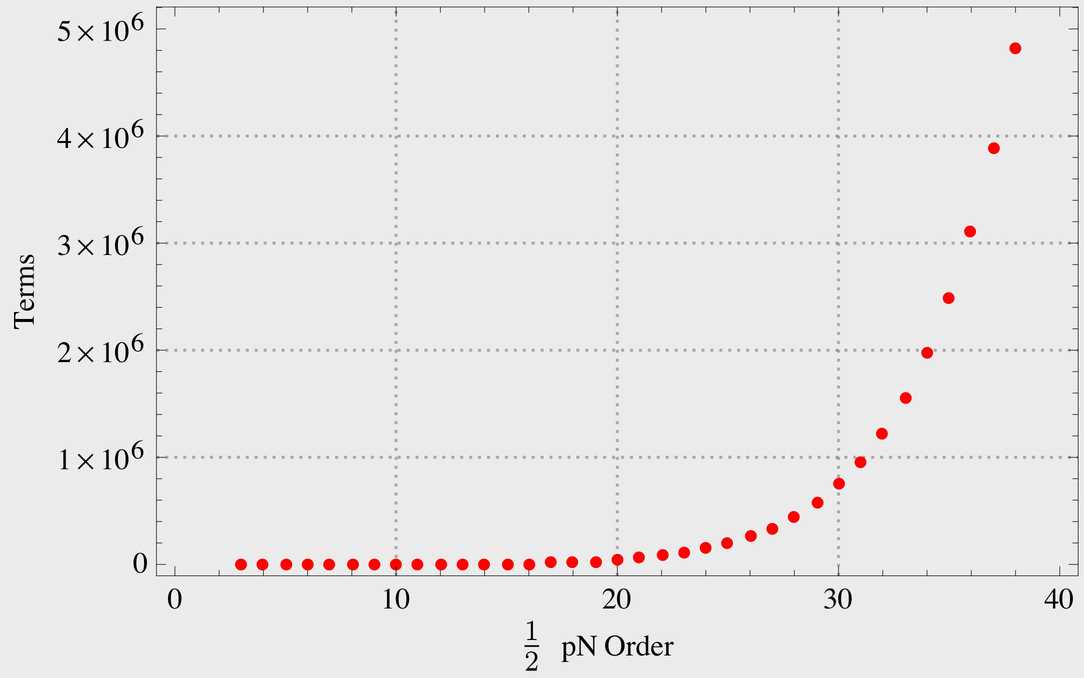
e.g.

$$b_3 = \frac{3ir^3\omega^3}{28(-1+a)^2(1+a)^2} + \omega\Bigg(- \frac{5a}{3(-1+a)^2(1+a)^2} + \frac{i\left(-7+6a^2+9a^4\right)}{2(-1+a)^2(1+a)^2\left(1+3a^2\right)} - \frac{3i\gamma}{(-1+a)^2(1+a)^2} - \frac{3\left(1-ia+8a^2-3ia^3+3a^4\right)\kappa}{2(-1+a)^2a(1+a)^2\left(1+3a^2\right)} - \frac{3i\psi^{(0)}\left(\frac{2ia}{\kappa}\right)}{(-1+a)^2(1+a)^2}\Bigg)$$

$${}_sR^{\text{in}}_{\ell m\omega} = e^{\psi^{\text{in}}}\, {}_s\hat{R}^{\text{in}}_{\ell m\omega} \qquad \psi^{\text{in}} = (\ldots)\frac{\omega}{c^3} + \ldots$$

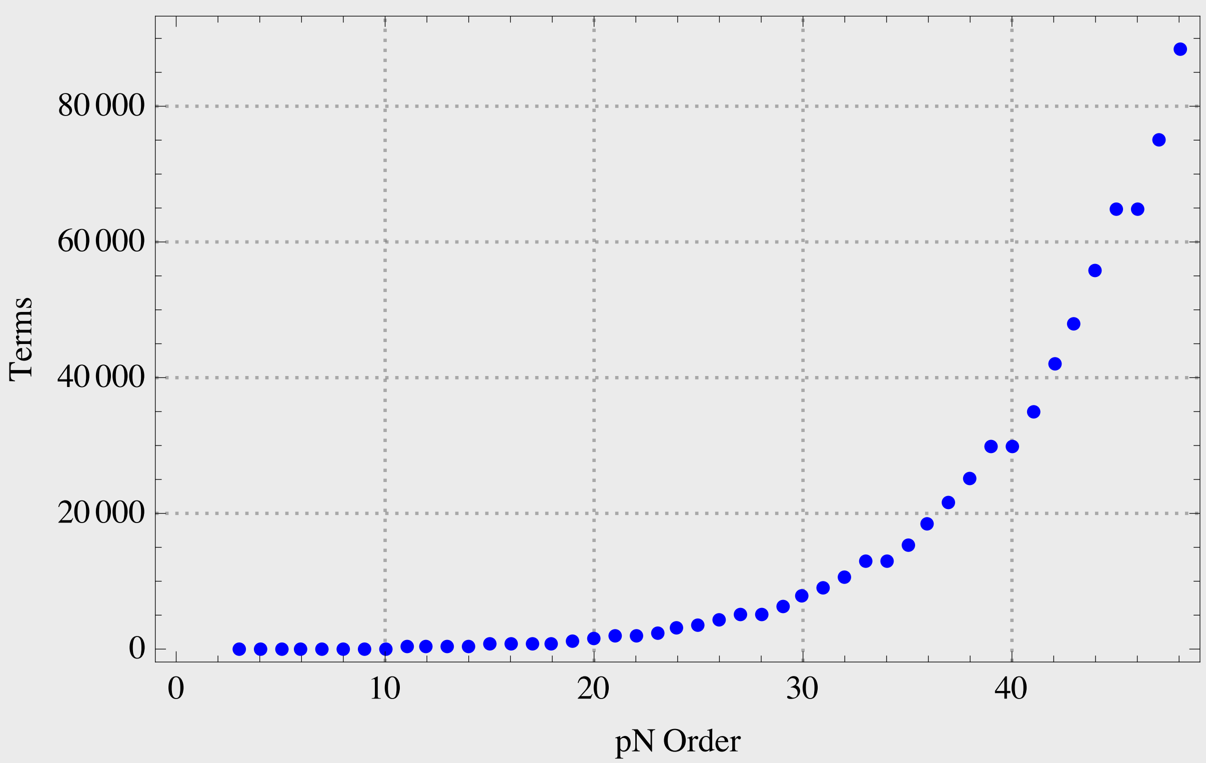
Analytics with MST series- large radius, small frequency

Before



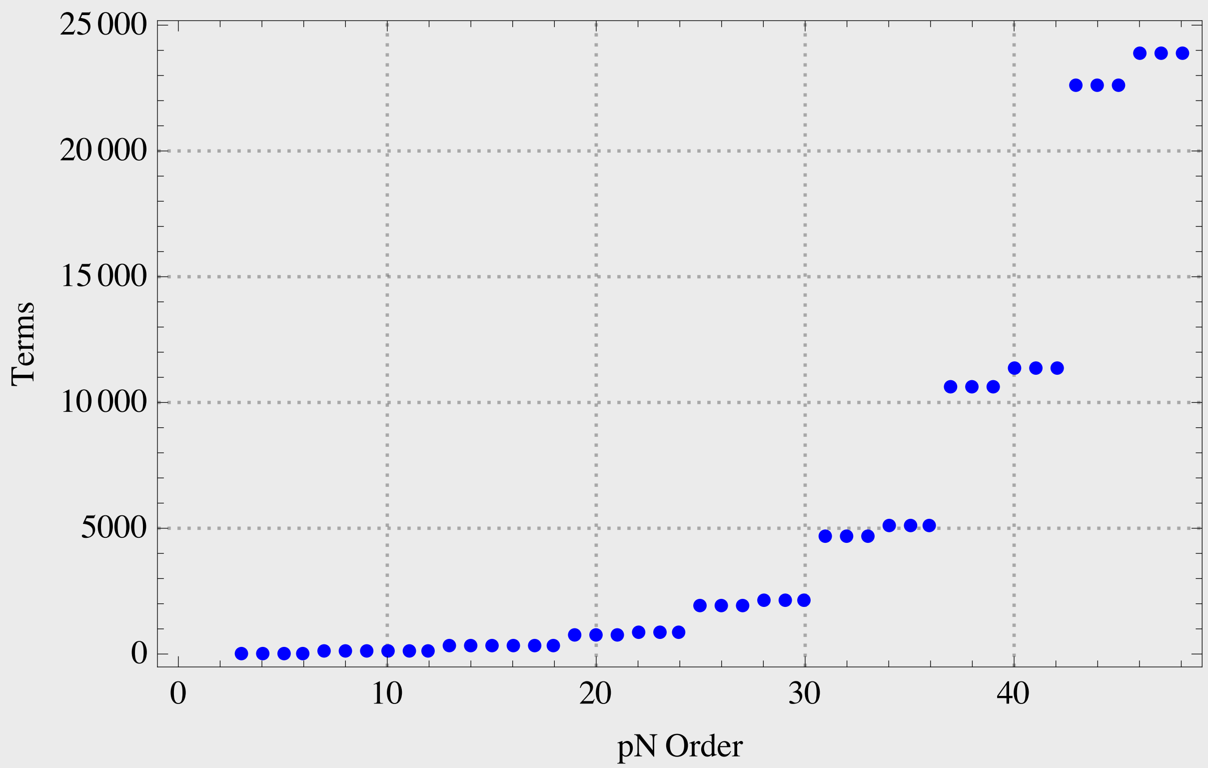
After

$_s\hat{R}^{\mathrm{in}}_{\ell m\omega}$



+

ψ^{in}



Analytics with MST series- large radius, small frequency

For 'large' ℓ the homogeneous solutions are simple

$$ {}_sR^{\rm up}_{\ell m\omega}(r) = r^{\nu+s}\left(1 + \frac{A_1^{\ell,m}}{c} + \frac{A_2^{\ell,m}}{c^2} + \ldots\right) $$

$$ {}_sR^{\rm in}_{\ell m\omega}(r) = r^{-\nu-1-s}\left(1 + \frac{B_1^{\ell,m}}{c} + \frac{B_2^{\ell,m}}{c^2} + \ldots\right) $$

e.g.

$A_1^{\ell,m} =$

$A_2^{\ell,m} =$

$A_3^{\ell,m} =$

- Use homogeneous solutions to reconstruct the metric via CCK
- expand explicitly in inverse radius

$$ \rightarrow h_{\mu\nu}^{\ell m,\rm ret} $$

-No more special functions of MST!

l mode regularisation with spheroidal harmonics

- Divergence appears in the infinite sum

$$H \equiv \frac{1}{2} h_{uu}$$

$$H^R = \sum_{\ell=0}^{\infty} (H_\ell^{ret} - H_\ell^S)$$

$$H_\ell^S = H_{[0]} + \frac{H_{[-2]}}{\lambda(\ell)} + ...$$

- HS is calculated using Detweiler-Whiting near the worldline

$$H^R = \sum_{\ell=0}^{\infty} (H_\ell^{ret} - H_{[0]})$$

l mode regularisation with spheroidal harmonics

- PN gives large l exactly

$$H_\ell^{\rm ret} = y - \frac{\left(21+2l+2l^2\right)y^2}{2(-1+2l)(3+2l)} + \frac{2\left(6+l+l^2\right)qy^{5/2}}{3l(1+l)} + \left(-\frac{3\left(960-1941l-1805l^2+298l^3+214l^4+78l^5+26l^6\right)}{8l(1+l)(-3+2l)(-1+2l)(3+2l)(5+2l)} - \frac{\left(3+l+l^2\right)q^2}{(-1+2l)(3+2l)}\right)y^3 + \ldots$$

- Read off reg params

$$H_{[0]}^{\rm Large\ell} = y - \frac{y^2}{4} + \frac{2}{3}qy^{5/2} + \left(-\frac{39}{64}-\frac{q^2}{4}\right)y^3 + \frac{7}{6}qy^{7/2} + \left(-\frac{385}{256}-\frac{7q^2}{36}\right)y^4 + \left(\frac{99q}{32}-\frac{q^3}{2}\right)y^{9/2} + \left(-\frac{61559}{16384}-\frac{1625q^2}{2304}+\frac{9q^4}{64}\right)y^5 + \left(\frac{3239q}{384}-\frac{733q^3}{648}\right)y^{11/2} + \ldots$$

$$H_{[-1]}^{\rm Large\ell} = 0$$

$$H_{[-2]}^{\rm Large\ell} = -\frac{45y^2}{16} + 4qy^{5/2} + \left(-\frac{759}{128}-\frac{15q^2}{16}\right)y^3 + \frac{59}{8}qy^{7/2} + \left(-\frac{16887}{1024}+\frac{17q^2}{48}\right)y^4 + \left(\frac{1989q}{64}-\frac{19q^3}{8}\right)y^{9/2} + \left(-\frac{757863}{16384}-\frac{40223q^2}{3072}+\frac{45q^4}{128}\right)y^5 + \left(\frac{53283q}{512}-\frac{587q^3}{288}\right)y^{11/2} + \ldots$$

l mode regularisation with spheroidal harmonics

$$h_{\mu\nu} = \sum_{\ell,m} h^{\ell m}_{\mu\nu} Y_{\ell m}(\alpha, \beta)$$

- Scalar sphericals, tensor harmonics, spin-weighted spherical/spheroidal
- Teukolsky+CCK-Complicated combination of spheroidal harmonics+derivatives

$h^S_{\mu\nu}$

Scalar spherical	Barack & Ori, Heffernan, Ottewill & Wardell
Tensor spherical	Wardell & Warburton 15'
Spheroidal	CK, Ottewill & Wardell 16’ (redshift inv)

l mode regularisation with spheroidal harmonics

- To use the scalar harmonic RP's, you need to change basis, e.g.

$$S_{\hat{\ell}m}(\sigma) = \sum_{\ell=|m|}^{\infty} b^{\ell}_{\hat{\ell}m}(\sigma) Y_{\ell m} \qquad b^{\ell}_{\hat{\ell}m}(\sigma) \to 0 \text{ as } |\hat{\ell} - \ell| >> 1$$

- Reordering sums you get couplings between modes e.g.

$$\begin{aligned} h^{ret,\hat{\ell}m}_{\mu\nu} S_{\hat{\ell}m} &= h^{ret,\hat{\ell}m}_{\mu\nu}(b^{\hat{\ell}}_{\hat{\ell}m} Y_{\hat{\ell}m} + b^{\hat{\ell}\pm 2}_{\hat{\ell},m} Y_{\hat{\ell}\pm 2,m} + \ldots) \\ &= (h^{ret,\hat{\ell}m}_{\mu\nu} b^{\hat{\ell}}_{\hat{\ell}m} + h^{ret,\hat{\ell}\mp 2,m}_{\mu\nu} b^{\hat{\ell}\mp 2}_{\hat{\ell}m} + \ldots) Y_{\hat{\ell}m} \end{aligned}$$

→

- Cumbersome for high order PN
- 'lose' ell modes for numerics

I mode regularisation with spheroidal harmonics

- Wardell & Warburton- Used Detweiler Whiting to get regularisation parameters for the entire metric and it's derivatives in a tensor spherical harmonic basis- importantly for PN

$$H^{\text{tensor}}_{[0]} = H^{\text{scalar}}_{[0]}$$

- Maybe also true for $H_{[0]}$ in the spheroidal basis..

$$H^{\text{large } \ell}_{[0]} = y - \frac{y^2}{4} + \frac{2}{3}qy^{5/2} - \left(\frac{39}{64} + \frac{q^2}{4}\right)y^3 + \frac{7}{6}qy^{7/2} - \left(\frac{385}{256} + \frac{7q^2}{36}\right)y^4 + \left(\frac{99q}{32} - \frac{q^3}{2}\right)y^{9/2} + \ldots$$

$$H^{\text{scalar}}_{[0]} = y - \frac{y^2}{4} + \frac{2}{3}qy^{5/2} - \left(\frac{39}{64} + \frac{q^2}{4}\right)y^3 + \frac{7}{6}qy^{7/2} - \left(\frac{385}{256} + \frac{47q^2}{288}\right)y^4 + \left(\frac{99q}{32} - \frac{q^3}{2}\right)y^{9/2} + \ldots$$

l mode regularisation with spheroidal harmonics

$$h^{S,\ell}_{\mu\nu} = \sum_{m=-\ell}^{\ell} h^{S,\ell m}_{\mu\nu} S_{\ell m}(\alpha, \beta, a\omega)$$

$$h^{S,\ell m}_{\mu\nu} = \int h^S_{\mu\nu} S^*_{\ell m}(\alpha, \beta, a\omega) d\Omega$$

using a local expansion of $h^S_{\mu\nu}$.We've learned in PN one can always do a low frequency expansion, so expand the harmonic for small $a\omega$

$$S_{\ell m}(\theta, \varphi; a\omega) = Y_{\ell m} + a^2\omega^2 \left[-\frac{\ell \alpha_{\ell-1}\alpha_\ell}{2\ell(2\ell-1)} Y_{\ell-2,m} + \frac{(\ell+1)\alpha_{\ell+1}\alpha_{\ell+2}}{2(\ell+1)(2\ell+3)} Y_{\ell+2,m} \right] + \mathcal{O}(a\omega^4)$$

and integrate term-by-term, using $\omega = m\Omega_\varphi$

l mode regularisation with spheroidal harmonics

$$h_{\mu\nu}^{S,\ell} = \sum_{m=-\ell}^{\ell} h_{\mu\nu}^{S,\ell m} S_{\ell m}(\alpha, \beta, a\omega)$$

$$h_{\mu\nu}^{S,\ell m} = \int h_{\mu\nu}^{S} S_{\ell m}^{*}(\alpha, \beta, a\omega) d\Omega$$

$$\mathcal{O}(1) : \sum_{m=-\ell}^{\ell} \left(\int h_{uu}^{S} Y_{\ell m}^{*} d\Omega \right) Y_{\ell m} \text{ - same as Scalar harmonic case}$$

$$\mathcal{O}(a\Omega_{\varphi}^{2}) : \sum_{m=-\ell}^{\ell} m \Bigg[\left(\int h_{uu}^{S} Y_{\ell-2,m}^{*} d\Omega + \int h_{uu}^{S} Y_{\ell+2,m}^{*} d\Omega \right) Y_{\ell m} + \int h_{uu}^{S} Y_{\ell m}^{*} d\Omega \left(Y_{\ell+2,m} + Y_{\ell-2,m} \right) \Bigg]$$

- factor of m gives extra powers of ℓ needed for a contribution to leading order regularisation parameter.

l mode regularisation with spheroidal harmonics

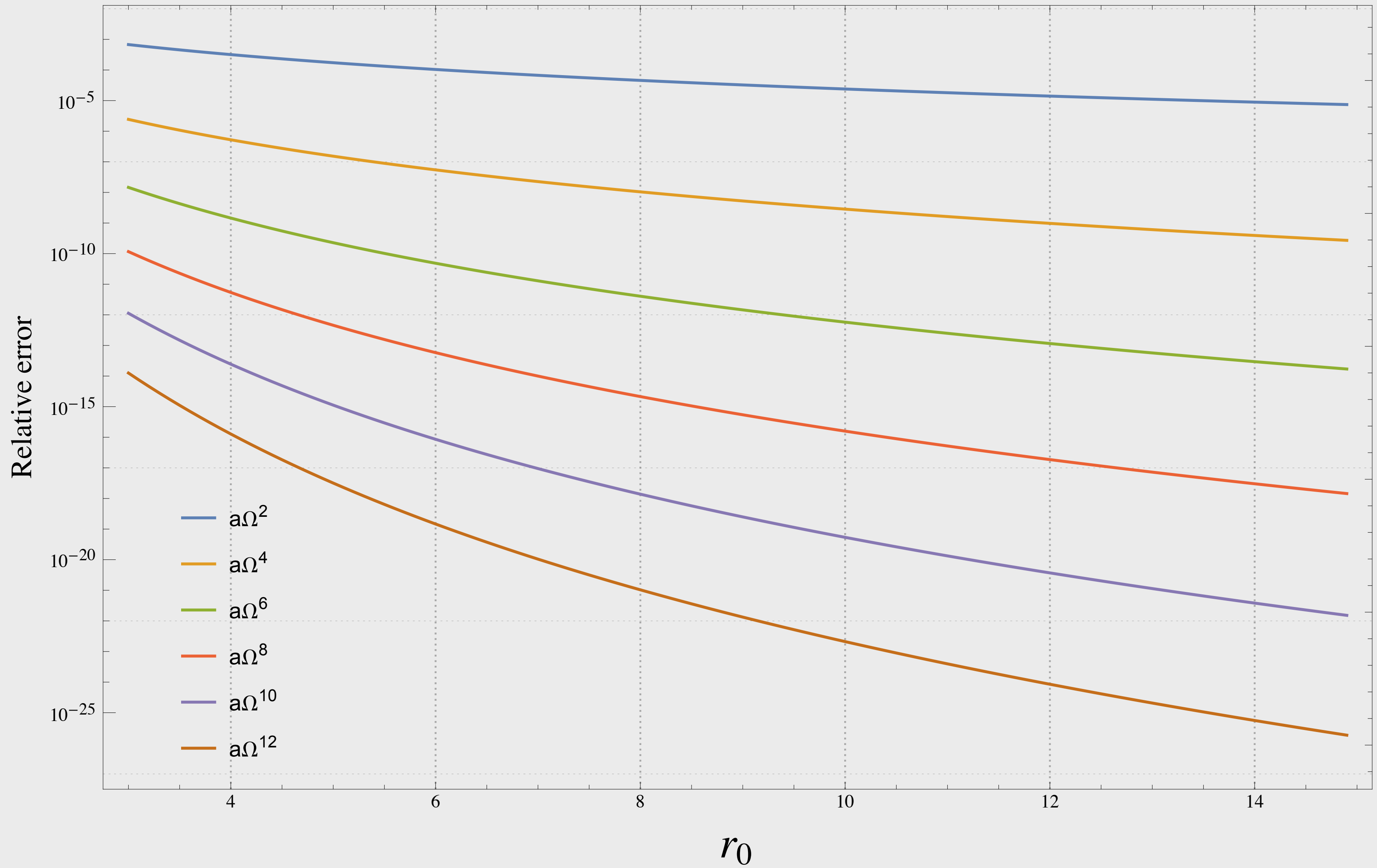
$$H_{[0]} = \sum_{i=0}^{6} c_i(r_0)(a\Omega)^{2i} + \mathcal{O}\left(\frac{a^{12}}{r_0^{22}}\right)$$

$$c_i \sim (\ldots)\mathcal{E} + (\ldots)\mathcal{K}$$

$$c_1 = \frac{(-2+k)\mathcal{E}(k)}{6\pi k^2\zeta} - \frac{(-1+k)\mathcal{K}(k)}{3\pi k^2\zeta}$$

Exact in r_0

Relative size of $a\Omega^n$ term for $a = .9$



l mode regularisation with spheroidal harmonics

Eccentric orbits should also be possible, comes down to the frequencies, e.g.

$$\sum_{m=-\ell}^{\ell}\sum_{n}(m\Omega_{\varphi}+n\Omega_{r})^{2}\Bigg[\left(\int h^{S}_{uu}Y^{*}_{\ell-2,m}d\Omega+\int h^{S}_{uu}Y^{*}_{\ell+2,m}d\Omega\right)Y_{\ell m}$$
$$+\int h^{S}_{uu}Y^{*}_{\ell m}d\Omega\,(Y_{\ell+2,m}+Y_{\ell-2,m})\Bigg]$$

- the radial frequency term can ‘probably’ be neglected in the leading order regularisation parameter.

Redshift invariant for circular orbits in Kerr -exact in spin parameter

$$\Delta U = c_1 y + c_2 y^2 + c_{2.5} y^{2.5} + c_3 y^3 + c_{3.5} y^{3.5} + c_4 y^4 + c_{4.5} y^{4.5} + (c_5 + c_5^{\ln} \log y) y^5 + c_{5.5} y^{5.5} + (c_6 + c_6^{\ln} \log y) y^6$$
$$\ldots + (c_{9.5} + c_{9.5}^{\ln} \log y + c_{9.5}^{\ln^2} \log^2 y + \mathcal{O}(y^{10})$$

$$c_1 = -1, \quad c_2 = -2, \quad c_{2.5} = \tfrac{7}{3} q, \quad c_3 = -5 - q^2, \quad c_{3.5} = \tfrac{46}{3} q, \quad c_4 = -\tfrac{121}{3} + \tfrac{41}{32}\pi^2 - \tfrac{86}{9} q^2, \quad c_{4.5} = 77q + q^3,$$
$$c_5 = -\tfrac{1157}{15} - \tfrac{128}{5}\gamma + \tfrac{677}{512}\pi^2 - \tfrac{256}{5}\log(2) - \tfrac{577}{9} q^2, \quad c_5^{\ln} = -\tfrac{64}{5}, \quad c_{5.5} = \left[\tfrac{974}{3} + \tfrac{29}{32}\pi^2\right] q + \tfrac{1526}{81} q^3,$$

$$c_{7.5} = \tfrac{81077}{3675}\pi + q\left[\tfrac{734961481}{22050} + \tfrac{2072}{5}\gamma - \tfrac{8911441}{3072}\pi^2 + \tfrac{4744}{7}\log(2) + \tfrac{972}{7}\log(3) + \tfrac{32}{5}\log(\kappa) + \tfrac{16}{5}\psi^{(0,2)}(q)\right]$$
$$+ q^3\left[\tfrac{243611}{225} + \tfrac{96}{5}\gamma + \tfrac{1319}{384}\pi^2 + \tfrac{96}{5}\log(2) + \tfrac{96}{5}\log(\kappa) + \tfrac{48}{5}\psi^{(0,2)}(q)\right] + \tfrac{12}{5} q^5,$$

$$\log(\sqrt{1-a^2}) \qquad \psi^{(n,k)}(q) \equiv 2\Re[\psi^{(n)}(\tfrac{ikq}{\kappa})]$$

http://www.barrywardell.net/research/code/mst-pn

Conclusion/Future

PN:

$\rightarrow$ Spin precession/tidal invariants?

$\rightarrow$ Exact in spin eccentric orbit redshift

$\rightarrow$ Generic orbits, a lot of progress in the dissipative sector

Spheroidal regularisation:

$\rightarrow$ Weyl scalars, these are pure spheroidal harmonics

$\rightarrow$ Details for more generic orbits/range of validity

$\rightarrow$ Higher order regularisation parameters